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Existence of Equilibrium and Zero-Beta Pricing Formula in the Capital Asset Pricing Model with Heterogeneous Beliefs*

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We study a mean-variance capital asset pricing model (CAPM) in which investors have different probability beliefs about assets returns and different attitudes towards risk, all assets are risky, short-selling is allowed and satiation is possible. First, we prove that there exists a competitive equilibrium in the model under a rather general condition. This condition indicates a simple relationship among initial endowment vectors, risk aversion ratio functions, perceived mean vectors and covariance matrices of all investors. Secondly, we derive a zero-beta pricing formula for the model which generalizes the well known Black's zero-beta pricing formula. In addition, we find in closed form an equilibrium price vector expressed in terms of perceived mean vectors, covariance matrices, and initial endowments of all investors. © 2003 Peking University Press

Key Words: Capital asset pricing model, Heterogeneity, Equilibrium theorem, Pricing formula.

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1. INTRODUCTION

The mean-variance capital asset pricing model (CAPM) with a riskless asset developed by Sharpe (1964), Lintner (1965) and Mossin (1966) has become a focal point in finance. The first significant extension of their work was made by Black (1972) who dropped the assumption of a riskless asset. This latter CAPM has been widely used in empirical work under the name of the "zero-beta" or two-factor CAPM. In these models it has been assumed that all investors' probability beliefs about assets returns are identical, and that each investor's expected utility is a function of the mean and variance of his portfolio return. Up to now, a great deal of research has been done on deriving various properties of equilibrium prices under a crucial assumption that an equilibrium does actually exist. The fundamental conclusions of these studies are the two (or mutual)-fund separation theorems and the "beta" pricing formula which indicate some simple relationship among the equilibrium prices of assets, the means, variances and covariances of their returns.

Initialized by Hart (1974), several equilibrium existence theorems have been established for the above classical CAPM by Nielsen (1989, 1990a-b), Allingham (1991), and Konno and Shirakawa (1995) among others. The current paper will provide sufficient conditions for the existence of equilibrium and furthermore derive an explicit pricing formula for a CAPM with many risky assets in which all investors' probability beliefs about assets returns are heterogeneous. Until now, little progress has been made with respect to CAPM with heterogeneous beliefs except for a special case. Namely, Nielsen (1990a) gives an existence condition for a CAPM in which investors may have different perceived covariance matrices but are all required to have the same perceived mean vector. One of the major moti-

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vations of the current paper comes from the concern and distrust about the assumptions of a homogeneous belief and a riskless asset in the standard CAPM. In his pioneering article (1964, pp.433-434) Sharpe makes the following statement:

"First, we assume a common pure rate of interest, with all investors able to borrow or lend funds on equal terms. Second, we assume homogeneity of investor expectations: investors are assumed to agree on the prospects of various investments–the expected values, standard deviations and correlation coefficients described in Part II. Needless to say, these are highly restrictive and undoubtedly unrealistic assumptions."

Later, Lintner (1969), Gonedes (1976) and Williams (1977) among others have voiced the same concern. Indeed, it is often observed in financial markets that different investors have different expectations (or beliefs) about assets returns and different attitudes towards risk. Of course, it is also equally true that many investors often have similar (but never the same) expectations about assets returns and attitudes towards risk. The results in this paper will somehow confirm and explain this common phenomenon.

To be precise, in this paper we will consider a mean-variance capital asset pricing model in which investors have different probability beliefs about assets returns and different attitudes towards risk, all assets are risky, short-selling is allowed and satiation is possible. Therefore, this model is much closer to many circumstances of financial exchange in the real world than the existing models. First, it will be shown (Theorem 2.3) via a novel fixed point argument that there exists at least one competitive equilibrium in the model under a rather general condition. This basic condition (Assumption 2.2) indicates a simple relationship among initial endowment vectors, risk aversion ratio functions, perceived mean vectors and covariance matrices of all investors. We also identifies several sufficient conditions for the existence of nonnegative equilibrium prices and derive several equilibrium theorems for homogeneous belief cases and constant risk aversion cases. Secondly, we derive a zero-beta valuation formula for the model which can be seen as a generalization of well known Black's zero-beta pricing formula. In the literature, research has been mostly directed on CAPM with a homogeneous belief in which all investors have the same perceived mean vector and covariance matrix and may have different initial endowment vectors. Even in this case, our basic condition turns out to be also weaker than those found by Allingham and Nielsen. Thus, their equilibrium results can be derived from ours as special cases. Furthermore, we prove that in the constant risk aversion case Allingham's condition (1991) is sufficient for the existence of equilibrium in a two risky assets economy in which investors have heterogeneous beliefs. But, it will be illustrated via an example that Allingham's condition is not sufficient for the existence of equilibrium in an economy with more than two risky assets.

We also refer to existence results established for incomplete market models by Duffie and Shafer (1985), Werner (1985), and Duffie (1987). Their results, in general, do not apply to the CAPM.

This paper is organized as follows. Section 2 sets up the two-period model, proves the main equilibrium existence results, and discusses the result of Allingham. Section 3 derives a zero-beta valuation formula for the model. Section 4 studies the constant risk aversion cases and derives the result of Nielsen. Finally, Section 5 gives a brief discussion of a dynamic extension of the two-period model.

2. THE MODEL AND ITS EQUILIBRIUM EXISTENCE THEOREM

Consider a two-period financial economy in which there are m investors, denoted by the set I_m, and n different assets, denoted by the set I_n. Each investor i is associated with a utility function $u^i : \mathbb{R}^n \mapsto \mathbb{R}$ and an initial endowment vector $w^i \in \mathbb{R}^n$. Let $\omega = \sum_{i=1}^m w^i$ denote the *market initial endowment.* It is assumed that $\omega \neq 0$. The utility function has a standard form: $u^i(x) = u^i(\rho, \sigma) = u^i(x^\top s^i, \frac{1}{2}x^\top T_i x)$, where $s^i \in \mathbb{R}^n$ is the perceived mean vector of investor i, and $T_i \in \mathbb{R}^{n\times n}$ is the perceived covariance matrix of investor i. In general, these matrices T_i and vectors s^i may differ cross investors. It is assumed that the matrix T_i is symmetric positive definite; u^i is a strictly concave C^1 function of x with

$$\frac{\partial u^i(\rho,\sigma)}{\partial \rho} > 0, \quad \frac{\partial u^i(\rho,\sigma)}{\partial \sigma} < 0.$$

Thus, u^i is a mean-variance utility function and is an increasing function of the expected return ρ and a decreasing function of the expected variance σ. Now we can define the *risk aversion ratio function* $r^i : \mathbb{R}^n \mapsto \mathbb{R}$ by

$$r^i(x) = -\frac{\frac{\partial u^i(\rho,\sigma)}{\partial \sigma}}{\frac{\partial u^i(\rho,\sigma)}{\partial \rho}}.$$

Clearly, r^i is a continuous function with positive values. Following from this, we know that all investors are risk averse. Such a financial economy will be denoted by

$$\mathcal{E} = \Big\langle (T_i, w^i, s^i, r^i),\, i \in I_m \Big\rangle.$$

Note that in this economy, because the utility function of each investor is not monotone, satiation is possible. Satiation refers to the situation where there is an optimal portfolio beyond which the increased return of holding more assets may not be sufficient to offset the increased risk. Such optimal portfolio is called a *satiation portfolio.* In fact, it is not difficult to show that in the economy $\mathcal{E}$ each investor i has a unique satiation portfolio $\bar{x}^i$; see Allingham (1991). That is, u^i achieves its unconstrained global maximum only at $\bar{x}^i$. Let $\bar{r}^i = r^i(\bar{x}^i)$. Then, by the first order condition, we see that $\bar{x}^i = \frac{T_i^{-1}s^i}{\bar{r}^i}$. Note that we do not require $w^i \geq 0$ or $s^i \geq 0$. So short-selling is allowed and some assets may have negative returns. Given $p \in \mathbb{R}^n$, the demand $x^i(p)$ of investor i is the maximizer of u^i under his budget constraint, and is unique. Following from the property of u^i, one can show that the set $D^i = \{x^i \in \mathbb{R}^n \mid u^i(x^i) \geq u^i(w^i)\}$ is a nonempty, convex and compact set. Let V^i be the set $\{r^i(x) \mid x \in D^i\}$ and let V be the product of all V^i, $i \in I_m$. Clearly, V is also a nonempty, convex and compact.

Note that some equilibrium properties with respect to simpler forms of the model described above have been studied by Lintner (1969), Gonedes (1976), Williams (1977), and Rabinovitch and Owen (1978).

The virtue of the mean-variance analysis lies in its simplicity and intuitive economic interpretations. Chamberlain (1983) and Owen and Rabinovitch (1983) have shown that mean-variance analysis is also consistent with expected utility maximization with general utility functions if the total returns follow a fairly large class of distribution patterns, including the normal distributions. Berk (1997) provides a set of conditions under which expected utility maximization and mean-variance analysis are equivalent.

In the sequel, we denote $(\frac{T_i^{-1}s^i}{\beta^i} - w^i)$ by $v^i(\beta^i)$ for each $i \in I_m$ and $\sum_{i \in I_m}(\frac{T_i^{-1}s^i}{\beta^i} - w^i)$ by $v(\beta)$ for $\beta = (\beta^1, \cdots, \beta^m) \in V$. $v^i(\beta^i)$ and $v(\beta)$ are called *generalized initial endowment of investor* i and *generalized market endowment,* respectively. Especially, let $\bar{v}^i = v^i(\bar{r}^i) = \bar{x}^i - w^i$ and $\bar{v} = v(\bar{r})$.

Definition 2.1. A vector $p^* \in \mathbb{R}^n \setminus \{0\}$ is an equilibrium price vector if there exist x^{1*}, x^{2*}, $\cdots$, $x^{m*} \in \mathbb{R}^n$ such that

(a) for each $i \in I_m$, x^{i*} is a solution of the problem

$$\text{maximize } u^i(x^i)$$

$$\text{subject to } \; p^{*\top}x^i \leq p^{*\top}w^i$$

(b) $\sum_{i \in I_m} x^{i*} = \sum_{i \in I_m} w^i$.

In the definition the vectors x^{1*}, $\cdots$, x^{m*} are called *equilibrium allocations.* It is well known from general equilibrium theory that there may fail to exist an equilibrium in the above economy, because the choice sets are unbounded below, because such sets contain satiation portfolios, and because there exist disagreements about assets returns among investors. See e.g. Hart(1974), Nielsen(1990b) and Allingham(1991). The fundamental question arises at once. Under what condition does the given economy have an equilibrium? In the following we will give a positive answer to this question.

For any positive integer k, we denote the $(k-1)$-dimensional unit simplex by

$$S^k = \{p \in \mathbb{R}^k_+ \mid \sum_{j=1}^{k} p_j = 1\}.$$

For every $\alpha \in S^m$, let

$$T(\alpha) = (\sum_{i \in I_m} \alpha_i T_i^{-1})^{-1}.$$

Such a $T(\alpha)$ is called a *market risk-measure matrix* with respect to α. Since each T_i is symmetric positive definite, it follows that $T(\alpha)$ is also symmetric positive definite.

In order for the economy to have an equilibrium, we impose the following basic condition on the economy $\mathcal{E}$.

ASSUMPTION 1. *For every $\alpha \in S^m$, every $i \in I_m$ and every $\beta \in V$, it holds that*

$$v(\beta) \neq 0 \quad and \quad v^{i\top}(\beta^i) T(\alpha) v(\beta) \geq 0.$$

This assumption says (i): at least one investor is not satiated; (ii): the angle between each investor's generalized initial endowment vector v^i and the generalized market endowment vector v with respect to the market risk-measure matrix $T(\alpha)$ should not be too big. Note that this relation does not depend on the magnitude of the vectors v^i and v. Loosely speaking, this assumption implies that investors' probability beliefs on assets returns can be completely different as long as they do not differ significantly. Finally, we point out that it will be hard to relax this assumption since it is in fact a necessary and sufficient condition in constant risk aversion cases (see Section 4). A simpler but stronger condition will be: For every $\alpha \in S^m$, every $i \in I_m$ and every $\beta \in V$, it holds that

$$v^{i\top}(\beta^i) T(\alpha) v(\beta) > 0.$$

Now we are ready to present our basic equilibrium existence theorem.

THEOREM 1. *Under Assumption 1, the economy $\mathcal{E} = \langle (T_i, w^i, s^i, r^i),\ i \in I_m \rangle$ has an equilibrium price vector.*

Proof. Let $P = \{p \in \mathbb{R}^n \mid \sum_{i=1}^n p_i^2 = 1\}$. In the economy $\mathcal{E}$, given $p \in P$, investor i has to solve the following decision problem

$$\text{maximize } u^i(x^i) = u^i(x^{i\top} s^i, x^{i\top} T_i x^i/2)$$

$$\text{subject to } \ p^\top x^i \leq p^\top w^i$$

Note that u^i is a strictly concave function. Therefore, if $p^\top \bar{v}^i = p^\top(\bar{x}^i - w^i) < 0$, the demand $x^i(p)$ of investor i is given by $x^i(p) = \bar{x}^i$; otherwise, i.e., $p^\top \bar{v}^i = p^\top(\bar{x}^i - w^i) \geq 0$, it follows from the Kuhn-Tucker condition that the demand $x^i(p)$ of investor i is given by

$$x^i(p) = \frac{T_i^{-1}(s^i - \gamma_i(p)p)}{r^i(x^i(p))}$$

where

$$\gamma_i(p) = \frac{p^\top(T_i^{-1}s^i - r^i(x^i(p))w^i)}{p^\top T_i^{-1} p} \geq 0.$$

In the sequel, we will simply use $r^i(p)$ and $r(p)$ to denote $r^i(x^i(p))$ and $(r^1(p), \cdots, r^m(p))$, respectively. Clearly, $(r^1(p), \cdots, r^m(p)) \in V$ for all $p \in P$, and $r^i(p) = \bar{r}^i$ when $p^\top \bar{v}^i \leq 0$. Therefore to show the existence of equilibrium in the economy $\mathcal{E}$, it is sufficient to show that there exists $p^* \in P$ with $p^{*\top}\bar{v}^i \geq 0$ for all $i \in I_m$, such that $\sum_{i \in I_m}(x^i(p^*) - w^i) = 0$, i.e.,

$$\sum_{i=1}^m \left(T_i^{-1}s^i/r^i(p^*) - w^i - \frac{p^{*\top}(T_i^{-1}s^i - r^i(p^*)w^i)}{r^i(p^*)p^{*\top}T_i^{-1}p^*} T_i^{-1}p^* \right) = 0$$

In order to achieve this goal, we construct a collection $\{\mathcal{E}(\alpha,\beta) \mid \alpha \in S^m, \beta \in V\}$ of new economies as follows. For each $\alpha \in S^m$ and each $\beta \in V$, define the economy

$$\mathcal{E}(\alpha,\beta) = \left\langle (T(\alpha), -v^i(\beta^i), 0, 1),\, i \in I_m \right\rangle.$$

Let $k(\alpha,\beta) > 0$ be a real number such that

$$p = k(\alpha,\beta)T(\alpha)v(\beta) \in P. \tag{1}$$

Given this p, investor i has to solve following decision problem

$$\text{minimize } \ x^{i\top} T(\alpha) x^i/2$$

$$\text{subject to } \; p^\top x^i \le p^\top(-v^i(\beta^i)).$$

The demand $x^i(\alpha,\beta)$ of investor i is given by

$$x^i(\alpha,\beta) = -\gamma_i T^{-1} p$$

provided that $\gamma_i \ge 0$. Assumption 1 and the proper choice of p imply that

$$\gamma_i = \frac{p^\top v^i(\beta^i)}{p^\top T^{-1}(\alpha)p} \ge 0.$$

Thus, the demand of investor i is

$$x^i(\alpha,\beta) = -\frac{p^\top v^i(\beta^i)}{p^\top T^{-1}(\alpha)p} T^{-1}(\alpha)p.$$

Therefore, the excess demand function is given by

$$z(\alpha,\beta) = v(\beta) - \frac{p^\top v(\beta)}{p^\top T^{-1}(\alpha)p} T^{-1}(\alpha)p.$$

It is clear that p is the unique solution of equation $z(\alpha,\beta) = 0$ in P. Obviously, p is a continuous function in α and β. So we write p as $p(\alpha,\beta)$. Recall that Assumption 1 implies

$$p^\top(\alpha,\beta)v^i(\beta^i) = k(\alpha,\beta)v^{i\top}(\beta)T(\alpha)v(\beta) \ge 0$$

for all $i \in I_m$. Moreover,

$$\sum_{i\in I_m} p^\top(\alpha,\beta)v^i(\beta^i) = p^\top(\alpha,\beta)v(\beta) = k(\alpha,\beta)v^\top(\beta)T(\alpha)v(\beta) > 0.$$

Hence $p^\top(\alpha,\beta)v^i(\beta^i) > 0$ for at least one $i \in I_m$. Thus we can define the continuous function $(\phi,\psi) : S^m \times V \mapsto S^m \times V$ by

$$(\phi(\alpha,\beta),\psi(\alpha,\beta)) = \\ \left(\frac{\frac{p^\top(\alpha,\beta)v^1(\beta^1)}{p^\top(\alpha,\beta)T_1^{-1}p(\alpha,\beta)}}{\sum_{i\in I_m}\frac{p^\top(\alpha,\beta)v^i(\beta^i)}{p^\top(\alpha,\beta)T_i^{-1}p(\alpha,\beta)}}, \cdots, \frac{\frac{p^\top(\alpha,\beta)v^m(\beta^m)}{p^\top(\alpha,\beta)T_m^{-1}p(\alpha,\beta)}}{\sum_{i\in I_m}\frac{p^\top(\alpha,\beta)v^i(\beta^i)}{p^\top(\alpha,\beta)T_i^{-1}p(\alpha,\beta)}}, \\ r^1(p(\alpha,\beta)), \cdots, r^m(p(\alpha,\beta)) \right).$$

Note that $S^m \times V$ is a convex and compact set and that (ϕ,ψ) is continuous. Then by Brouwer's fixed point theorem there exists $(\alpha^*,\beta^*) \in S^m \times V$ such that $(\alpha^*, \beta^*) = (\phi(\alpha^*, \beta^*), \psi(\alpha^*, \beta^*))$. Let $p^* = p(\alpha^*, \beta^*)$. Then we have $\beta^* = (\beta^{1*}, \cdots, \beta^{m*}) = r(p^*)$. The definition of $r^i(p)$ tells us that $p^{*\top}\bar{v}^i > 0$ when $r^i(p^*) \neq \bar{r}^i$. On the other hand, if $r^i(p^*) = \bar{r}^i$, then $v^i(r^i(p^*)) = \bar{v}^i$ and $p^{*\top}\bar{v}^i = p^\top(\alpha^*, \beta^*)v^i(r^i(p^*)) = p^\top(\alpha^*, \beta^*)v^i(\beta^{i*}) \geq 0$. Hence we have that $p^{*\top}\bar{v}^i \geq 0$ for all $i \in I_m$. Moreover, note that

$$z(\phi(\alpha^*, \beta^*), \psi(\alpha^*, \beta^*)) = z(\alpha^*, \beta^*) = 0$$

with solution p^*. That is to say,

$$\begin{aligned} 0 &= z(\phi(\alpha^*, \beta^*), \psi(\alpha^*, \beta^*)) \\ &= v(r(p^*)) - \tfrac{p^{*\top} v(r(p^*))}{p^{*\top} T^{-1}(\phi(\alpha^*, \beta^*))p^*} T^{-1}(\phi(\alpha^*, \beta^*))p^* \\ &= v(r(p^*)) - \textstyle\sum_{i=1}^m \tfrac{p^{*\top} v^i(r^i(p^*))}{p^{*\top} T_i^{-1} p^*} T_i^{-1} p^* \\ &= \textstyle\sum_{i=1} \left(\tfrac{T_i^{-1} s^i}{r^i(p^*)} - w^i - \tfrac{p^{*\top}(T_i^{-1} s^i - r^i(p^*) w^i)}{r^i(p^*) p^{*\top} T_i^{-1} p^*} T_i^{-1} p^* \right). \end{aligned}$$

Obviously, p^* is also an equilibrium price vector for the original economy $\mathcal{E}$. We are done. ∎

This result somehow gives a theoretic explanation of a common phenomenon in financial markets of the real world that even if investors often have different expectations about market returns, markets are still in equilibrium at most of time and market failure or collapse does occur but not so often. It might be also worth mentioning that the equilibrium price vector p^* in the above proof can be efficiently computed by using fixed point methods; see Yang (1999a). This means that there exists at least one computable equilibrium in the economy $\mathcal{E}$. From a practical point of view, this computable property is quite desirable.

Now we will identify two sufficient conditions for the existence of a nonnegative equilibrium price vector. In the above theorem, if P happens to be a subset of $\mathbb{R}^n_+$, then we have a nonnegative equilibrium price vector.

Corollary 1. *Let $\mathcal{E} = \left\langle (T_i, w^i, s^i, r^i),\, i \in I_m \right\rangle$ be an economy such that $v^i(\beta^i) >> 0$ for all $i \in I_m$ and $T(\alpha)v(\beta) >> 0$ for all $\alpha \in S^m$ and $\beta \in V$. Then the economy has at least one equilibrium price vector $p^* \in \mathbb{R}^n_+ \setminus \{0\}$.*

Corollary 2. *Let $\mathcal{E} = \left\langle (T_i, w^i, s^i, r^i),\, i \in I_n \right\rangle$ be an economy such that for all $i \in I_m$, $\beta^i w^i \ll T_i^{-1} s^i$ for all $\beta \in V$ and all off-diagonal elements of T_i^{-1} are nonpositive. Then the economy has at least one equilibrium price vector $p^* \in \mathbb{R}^n_+ \setminus \{0\}$.*

Proof. Since T_i is positive definite, it is easy to verify that all diagonal elements of T_i^{-1} are positive and T_i^{-1} is also positive definite. Thus T_i^{-1} satisfies all conditions of Hawkins-Simon theorem (1949). That means that for any $c \in \mathbb{R}^n_+$, the equation $T_i^{-1}x = c$ has a unique nonnegative solution. For each $\alpha \in S^m$, let $B = T^{-1}(\alpha)$. Clearly, all off-diagonal elements of B are nonpositive, all diagonal elements of B are positive and B is positive definite. Therefore, B satisfies all conditions of Hawkins-Simon theorem. That means that for any $c \in \mathbb{R}^n_+$, $Bx = c$ has a unique nonnegative solution x and thus $T(\alpha)c$ is nonnegative. Now the conclusion follows immediately. ∎

Finally we turn to discuss the CAPM with a homogeneous belief. That is, all investors have the same perceived mean vector s and covariance matrix T. The following result is the main equilibrium theorem (i.e., Theorem 1) of Allingham (1991). We point out that in his Theorem 1 the assumption [1] may not be sufficient but can be corrected as follows.

THEOREM 2. *Let $\mathcal{E} = \left\langle (T, w^i, s, r^i),\ i \in I_m \right\rangle$ be an economy such that $\beta^i w^i << T^{-1}s$ and $\beta^i T w^i << s$ for each $i \in I_m$ and each $\beta \in V$. Then the above economy has at least one equilibrium price vector $p^* \in \mathbb{R}^n_+ \setminus \{0\}$.*

Proof. Notice that for all $\alpha \in S^m$ we have $T(\alpha) = T$. For each i, it follows from $\beta^i w^i << T^{-1}s$ and $\beta^i T w^i << s$ for all $i \in I_m$ that

$$\begin{aligned} & (T^{-1}s/\beta^i - w^i)^\top T(\sum_{h \in I_m} (T^{-1}s/\beta^h - w^h)) \\ = & (T^{-1}s/\beta^i - w^i)^\top (\sum_{h \in I_m} (s/\beta^h - Tw^h)) > 0 \end{aligned}$$

by noting that $\beta^i > 0$ for all $i \in I_m$. Thus by Theorem 1 there exists an equilibrium price vector p^* for $\mathcal{E}$. $p^* > 0$ can be derived from equation 1, since

$$p(\alpha) = k(\beta)T(\sum_{h \in I_m} (T^{-1}s/\beta^h - w^h)) = k(\beta)(\sum_{h \in I_m} (s/\beta^h - Tw^h)) > 0$$

for all $\alpha \in S^m$. ∎

[1] More precisely, Allingham's $(\sup r^t(x))w^t \ll T^{-1}s$ condition may not be sufficient to ensure that $r^t(x)w^t << T^{-1}s$ for any $x \in A_t$. Thus, $\lambda_t < 0$ might happen in his proof. However, if $T^{-1}s \geq 0$, then his condition is sufficient and in fact coincides with the above one.

By applying Farkas' lemma, Konno and Shirakawa (1995) prove the existence of an equilibrium in a CAPM with a homogeneous belief in the presence of a riskless asset. The presence of a riskless asset rules out the possibility of satiation and thus makes their model easier to deal with.

3. A UNIVERSAL ZERO-BETA PRICING FORMULA

In this section we will derive a zero-beta pricing formula for CAPM with heterogeneous beliefs which generalizes the well-known zero-beta pricing formula derived by Black (1972) for CAPM with a homogeneous belief. This formula might be useful and important for empirical studies of such models.

For $\alpha, \gamma \in S^m$, let

$$s(\alpha, \gamma) = \sum_{i=1}^{m} \gamma_i T(\alpha) T_i^{-1} s^i.$$

We call $s(\alpha, \gamma)$ a *market expected return vector* with respect to α and γ. Now we have the following lemma showing a relationship among equilibrium price vector, market risk-measure matrix, market expected return vector and market initial endowment.

LEMMA 1. *Let the economy* $\mathcal{E} = \left\langle (T_i, w^i, s^i, r^i),\ i \in I_m \right\rangle$ *satisfy Assumption 1. Then for any equilibrium price vector* p*, there exist* $\alpha, \gamma \in S^m$*,* $\delta > 0$ *and* $\eta > 0$ *such that*

$$p = \delta[s(\alpha, \gamma) - \eta T(\alpha)\omega].$$

Proof. Following from Theorem 1 there exists an equilibrium $(p, x^1, \cdots, x^m)$. Let $c_i = \frac{\partial u^i(\rho,\sigma)}{\partial \rho} > 0$ and $d_i = \frac{\partial u^i(\rho,\sigma)}{\partial \sigma} < 0$. Then we have $\nabla u^i(x^i) = c_i s^i + d_i T_i x^i$. Since $(p, x^1, \cdots, x^m)$ is an equilibrium, the Kuhn-Tucker condition implies that

$$\lambda_i p = \nabla u^i(x^i) = c_i s^i + d_i T_i x^i, \tag{2}$$

where $\lambda_i \geq 0$ for all $i \in I_m$. Since at least one investor is not satiated, $\nabla u^i(x^i) \neq 0$ and $\lambda_i > 0$ for some i. Following from equation (2), one has

$$x^i = -\frac{c_i}{d_i} T_i^{-1} s^i + \frac{\lambda_i}{d_i} T_i^{-1} p.$$

Summing over i yields

$$\sum_{i=1}^{m} x^i = \omega = -\sum_{i=1}^{m} \frac{c_i}{d_i} T_i^{-1} s^i + \sum_{i=1}^{m} \frac{\lambda_i}{d_i} T_i^{-1} p.$$

Setting $\beta = -\sum_{i=1}^{m} \frac{\lambda_i}{d_i} > 0$, $\alpha_i = -\frac{\lambda_i}{\beta d_i} \geq 0$, then $\alpha = (\alpha_1, \cdots, \alpha_m) \in S^m$. Following from the above equation, one has

$$p = \frac{1}{\beta}[-\sum_{i=1}^{m} \frac{c_i}{d_i} T(\alpha) T_i^{-1} s^i - T(\alpha)\omega].$$

Letting $\eta_1 = -\sum_{i=1}^{m} \frac{c_i}{d_i} > 0$, $\gamma_i = -\frac{1}{\eta_1 b_i} \geq 0$, then $\gamma = (\gamma_1, \cdots, \gamma_m) \in S^m$. Setting $\eta = \frac{1}{\eta_1}$, and $\delta = \frac{1}{\eta\beta} > 0$, then one has

$$p = \delta[s(\alpha, \gamma) - \eta T(\alpha)\omega].$$

∎

Now we will derive a valuation formula for CAPM with heterogeneous beliefs. Recall that $\omega \neq 0$. Denote the mean and variance of return to the market initial endowment ω by $E_M = \omega^\top s(\alpha, \gamma)$ and $\sigma_M^2 = \omega^\top T(\alpha)\omega$. Let z be a "zero-covariance" portfolio, i.e., a portfolio with $p^\top z = p^\top \omega$ and $z^\top T(\alpha)\omega = 0$.

Theorem 3. *Suppose p is an equilibrium price vector. If x is a portfolio with $p^\top x = p^\top \omega$, then*

$$x^\top s(\alpha, \gamma) = z^\top s(\alpha, \gamma) + \frac{x^\top T(\alpha)\omega}{\sigma_M^2}[E_M - z^\top s(\alpha, \gamma)].$$

Proof. It follows from Lemma 1 that if y is a portfolio with $p^\top y = p^\top \omega$, then

$$\delta(y^\top s(\alpha, \gamma) - \eta y^\top T(\alpha)\omega) = p^\top y = p^\top \omega = \delta(E_M - \eta\sigma_M^2),$$

so that $y^\top s(\alpha, \gamma) - \eta y^\top T(\alpha)\omega = E_M - \eta\sigma_M^2$. In particular, $z^\top s(\alpha, \gamma) = E_M - \eta\sigma_M^2$, and $x^\top s(\alpha, \gamma) - \eta x^\top T(\alpha)\omega = z^\top s(\alpha, \gamma)$. Thus,

$$\begin{aligned} x^\top s(\alpha, \gamma) &= z^\top s(\alpha, \gamma) + \eta x^\top T(\alpha)\omega \\ &= z^\top s(\alpha, \gamma) + \tfrac{x^\top T(\alpha)\omega}{\sigma_M^2} \eta\sigma_M^2 \\ &= z^\top s(\alpha, \gamma) + \tfrac{x^\top T(\alpha)\omega}{\sigma_M^2}[E_M - z^\top s(\alpha, \gamma)]. \end{aligned}$$

$\blacksquare$

The above formula is not only first but also "fundamental", since, as you will see, the following Black's zero-beta pricing formula (see also Nielsen (1990b)) follows immediately. Note that since $T_i = T$ and $s^i = s$ for all $i \in I_m$, we have $T(\alpha) = T$ and $s(\alpha, \gamma) = s$.

COROLLARY 3. *Suppose p is an equilibrium price vector. If x is a portfolio with $p^\top x = p^\top \omega$, then*

$$x^\top s = z^\top s + \frac{x^\top T\omega}{\sigma_M^2}(E_M - z^\top s),$$

where $E_M = \omega^\top s$, $\sigma_M^2 = \omega^\top T\omega$, z is a zero-covariance portfolio with $p^\top z = p^\top \omega$ and $z^\top T\omega = 0$.

4. CONSTANT RISK AVERSION CASES

In the constant risk aversion cases, we have the following results which are specifications of Theorem 1. See also Yang (1999b).

THEOREM 4. *Let $\mathcal{E} = \left\langle (T_i, w^i, s^i, r^i),\ i \in I_m \right\rangle$ be an economy where r^i is a positive constant for each $i \in I_m$. If*

$$\sum_{h\in I_m}(\frac{T^{-1}s^h}{r^h} - w^h) \neq 0 \quad \text{and} \quad (\frac{T_i^{-1}s^i}{r^i} - w^i)^\top T(\alpha)\sum_{h\in I_m}(\frac{T_h^{-1}s^h}{r^h} - w^h) \geq 0$$

for all $i \in I_m$ and for all $\alpha \in S^m$, then the economy has at least one equilibrium price vector $p^* \in P$.

The next result is the main equilibrium theorem (i.e., Proposition 3) of Nielsen (1990b) in which homogeneous belief cases are considered, i.e., $T_i = T$ and $s^i = s$ for all i.

THEOREM 5. *Let $\mathcal{E} = \left\langle (T, w^i, s, r^i),\ i \in I_m \right\rangle$ be an economy where r^i is a positive constant for each $i \in I_m$. If*

$$\sum_{h\in I_m}(\frac{T^{-1}s}{r^i} - w^i) \neq 0 \quad \text{and} \quad (\frac{T^{-1}s}{r^i} - w^i)^\top \sum_{h\in I_m}(\frac{s}{r^h} - Tw^h) \geq 0$$

for all $i \in I_m$, then the economy has at least one equilibrium price vector $p^* \in P$.

In addition Nielsen shows that the above condition is also a necessary condition and the economy has only one equilibrium.

The following theorem shows that Allingham's condition is sufficient for the existence of equilibrium in an two-asset economy where investors have heterogeneous beliefs.

THEOREM 6. *Suppose that there are only two assets (i.e., $n = 2$) in an economy $\mathcal{E} = \left\langle (T_i, w^i, s^i, 1),\ i \in I_m \right\rangle$, and that $w^i \ll T_i^{-1} s^i$ and $T_i w^i \ll s^i$ for all $i \in I_m$. Then there exists an equilibrium price vector $p^* \in \mathbb{R}^2_+ \setminus \{0\}$ in this economy.*

Proof. Note that u^i is a strictly concave function. By Kuhn-Tucker condition, for any given $p \in R^n_+ \setminus \{0\}$, the demand $x^i(p)$ of investor i is given by

$$x^i(p) = T_i^{-1}(s^i - \beta_i(p)p)$$

provided that $\beta_i(p) \geq 0$. By the assumption that T_i is symmetric positive definite and $T_i^{-1} s^i \geq w^i$, we know

$$\beta_i(p) = \frac{p^\top (T_i^{-1} s^i - w^i)}{p^\top T_i^{-1} p} > 0.$$

Now consider the excess demand function $z(p) = \sum_{i \in I_m} (x^i(p) - w^i)$ defined on the unit simplex S^2. It is clear that $p^\top z(p) = 0$ for every $p \in S^2$. In other words, Walras' law holds. Now we will demonstrate that for each $p \in S^2$, $p_j = 0$ implies $z_j(p) \geq 0$. In fact, we shall prove a stronger result that $p_j = 0$ implies $x^i_j(p) - w^i_j \geq 0$ for all $i \in I_m$. Then we know that there exists a point $p^* \in S^2$ such that $z(p^*) = 0$. We are done. For simplicity, we ignore the indices of investors. So, let $s = (s_1, s_2)^\top$, $w = (w_1, w_2)^\top$ and

$$T = \begin{bmatrix} a & b \\ b & d \end{bmatrix}.$$

Then we have $a > 0$, $c > 0$, $ac - b^2 > 0$, and

$$T^{-1} = \begin{bmatrix} \frac{c}{ac-b^2} & \frac{-b}{ac-b^2} \\ \frac{-b}{ac-b^2} & \frac{a}{ac-b-2} \end{bmatrix}.$$

In case $p = (0, 1)^\top$, we have

$$\beta(p) = \frac{as_2 - bs_1 - w_2(ac - b^2)}{a}$$

and

$$z_1(p) = \frac{s_1 - aw_1 - bw_2}{a}.$$

By the assumption that $s \geq Tw$, it is easy to see that $z_1(p) \geq 0$. Similarly, in case $p = (1,0)^\top$, we have

$$\beta(p) = \frac{cs_1 - bs_2 - w_1(ac - b^2)}{c}$$

and

$$z_2(p) = \frac{s_2 - bw_1 - cw_2}{c} \geq 0.$$

▮

It is very tempting to think that Allingham's condition may be also sufficient for the existence of equilibrium in an economy with more than two risky assets. The following counter example shows unfortunately that it is not true any more. This example will be quite instructive in helping us understand why heterogeneous beliefs may cause market failure.

Example 4.1 For each positive integer k, define an economy

$$\mathcal{E}(k) = \Big\langle (T_i, w^i(k), s^i, r^i),\ i = 1,2,3,4,5 \Big\rangle$$

where $r^i = 1$, $s^i = (0,0,0)^\top$ for all i, $w^1(k) = w^2(k) = w^1 = w^2 = (-10, -100, -100)^\top$, $w^3(k) = (-\frac{1}{k}, -\frac{1}{k^2}, -\frac{1}{k^2})^\top$, $w^4(k) = (-\frac{1}{k^2}, -\frac{1}{k}, -\frac{1}{k^2})^\top$, $w^5(k) = (-\frac{1}{k^2}, -\frac{1}{k^2}, -\frac{1}{k})^\top$,

$$T_1 = \begin{bmatrix} 8 & -1 & 2 \\ -1 & 1 & 0 \\ 2 & 0 & 4 \end{bmatrix}, \quad T_2 = \begin{bmatrix} 8 & 2 & -1 \\ 2 & 4 & 0 \\ -1 & 0 & 1 \end{bmatrix},$$

and

$$T_3 = T_4 = T_5 = \mathbf{I} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

One can easily check that every $\mathcal{E}(k)$ satisfies Allingham's conditions, i.e., $w^i \ll T_i^{-1} s^i = 0$ and $T_i w^i \ll s^i = 0$. Nevertheless, we will show that there exists some k such that $\mathcal{E}(k)$ has no equilibrium at all. This will be done in two steps. Let $B^3 = \{x \in R^3 \mid x_1^2 + x_2^2 + x_3^2 = 1\}$.

Step 1: Consider the two-investor economy

$$\mathcal{E} = \Big\langle (T_i, w^i, s^i, r^i),\ i = 1, 2 \Big\rangle$$

where T_i, w^i, s^i and r^i are the same as in $\mathcal{E}(k)$ for $i = 1$ and 2. It is easy to check that the economy $\mathcal{E}$ satisfies all the conditions of Theorem 4 and so has an equilibrium. In fact we will demonstrate that there exists a unique equilibrium price $p^* = (p_1^*, p_2^*, p_3^*)^\top$ in the economy $\mathcal{E}$. First, note that

$$T_1^{-1} = \frac{1}{24}\begin{bmatrix} 4 & 4 & -2 \\ 4 & 28 & -2 \\ -2 & -2 & 7 \end{bmatrix}, \quad T_2^{-1} = \frac{1}{24}\begin{bmatrix} 4 & -2 & 4 \\ -2 & 7 & -2 \\ 4 & -2 & 28 \end{bmatrix}.$$

For each $\alpha \in [0,1]$, let $T(\alpha) = [\alpha T_1^{-1} + (1-\alpha)T_2^{-1}]^{-1}$. Now we consider the economy

$$\mathcal{E}(\alpha) = \Big\langle (T(\alpha), w^i, s^i, r^i),\ i = 1, 2 \Big\rangle.$$

Note that every equilibrium price $p(\alpha)$ of $\mathcal{E}(\alpha)$ must satisfy

$$T^{-1}(\alpha)p(\alpha) = -r(w^1 + w^2)$$

for some $r > 0$. Solving the above equation, we obtain

$$\begin{aligned} p_1(\alpha) &= 231\alpha^2 - 231\alpha + 48, \\ p_2(\alpha) &= -66\alpha^2 - 22\alpha + 112, \\ p_3(\alpha) &= -66\alpha^2 + 154\alpha + 24. \end{aligned}$$

It is easy to verify that $p^\top(\alpha)w^1 = p^\top w^2 < 0$ for all $\alpha \in [0,1]$. Note that a vector p is an equilibrium price vector of $\mathcal{E}$ if and only if there exists some $\alpha \in [0,1]$ such that $p = p(\alpha)$ and

$$\alpha = \frac{\frac{p^\top w^1}{p^\top T_1^{-1} p}}{\frac{p^\top w^1}{p^\top T_1^{-1} p} + \frac{p^\top w^2}{p^\top T_2^{-1} p}} = \frac{\frac{1}{p^\top T_1^{-1} p}}{\frac{1}{p^\top T_1^{-1} p} + \frac{1}{p^\top T_2^{-1} p}}.$$

Equivalently, there exists some $\alpha \in [0,1]$ such that $p = p(\alpha)$ and α is a solution of the following equation:

$$p^\top(\alpha)(\alpha T_1^{-1} + (1-\alpha)T_2^{-1})p(\alpha) = 0.$$

This leads to

$$\frac{11(2\alpha - 1)}{3}[3 \times 33^2\alpha^2(1-\alpha)^2 - 21 \times 88\alpha(1-\alpha) + 2^{10}] = 0,$$

where $\alpha \in [0,1]$. This equation has a unique solution $\alpha = \frac{1}{2}$, since the function $f(x) = 33 \times 99x^2 - 21 \times 88x + 2^{10}$ has a unique minimizer 28/99 with an approximate minimum 762.73. Therefore we obtained a unique equilibrium vector $p^* = \frac{1}{\sqrt{1361}}(-3, 26, 26)^\top$ in the economy $\mathcal{E}$.

Step 2: Suppose to the contrary that there exists at least one equilibrium price $p(k)$ in $\mathcal{E}(k)$ for each k. We can always choose $p(k)$ from B^3 for all k. Then there exists a convergent subsequence of $\{p(k) \mid k = 1, 2, \cdots, \}$. Without loss of generality, we may simply assume that $\lim_{k\to\infty} p(k) = \bar{p}$.

On the one hand, note that $p^\top(k)w^i(k) \le 0$ for $i = 1, 2, 3, 4, 5$. In other words, we have

$$\begin{aligned} p_1(k)/k + p_2(k)/k^2 + p_3(k)/k^2 &\ge 0, \\ p_1(k)/k^2 + p_2(k)/k + p_3(k)/k^2 &\ge 0, \\ p_1(k)/k^2 + p_2(k)/k^2 + p_3(k)/k &\ge 0. \end{aligned}$$

This implies that $p(k) \ge -2(1/k, 1/k, 1/k)^\top$ for all positive integers k. Furthermore, it is easy to see that the limit $\bar{p}$ of $p(k)$'s must be nonnegative. That is, $\bar{p} \ge 0$.

On the other hand, note that every equilibrium $p(k)$ of $\mathcal{E}(k)$ satisfies

$$\begin{aligned} &\sum_{i=1}^{5} \frac{p^\top(k)w^i(k)}{p^\top(k)T_i^{-1}p(k)} T_i^{-1}p(k) \\ = &\sum_{i=1}^{5} w^i = -(20 + \frac{1+2k}{k^2}, 200 + \frac{1+2k}{k^2}, 200 + \frac{1+2k}{k})^\top \end{aligned} \tag{3}$$

Since $p^\top T_i^{-1} p$ has a positive lower bound on B^3 for all i, we have

$$\lim_{k\to\infty} \frac{p^\top(k)w^i(k)}{p^\top(k)T_i^{-1}p(k)} = 0$$

for $i = 3$, 4 and 5. By taking the limit from both sides of equation (3), we have

$$\frac{\bar{p}^\top w^1}{\bar{p}^\top T_1^{-1}\bar{p}} T_1^{-1}\bar{p} + \frac{\bar{p}^\top w^2}{\bar{p}^\top T_2^{-1}\bar{p}} T_2^{-1}\bar{p} = -(20, 200, 200)^\top = w^1 + w^2.$$

Consequently, $\bar{p}$ must be an equilibrium of the economy $\mathcal{E}$. That is,

$$\bar{p} = p^* = \frac{1}{\sqrt{1361}}(-3, 26, 26)^\top,$$

yielding a contradiction to $\bar{p} \ge 0$.

Therefore we can conclude that there exists no equilibrium in $\mathcal{E}(k)$ for some k.

As a matter of fact, if we take $k \geq 100$, then the economy $\mathcal{E}(k)$ will not satisfy the condition of Theorem 4. This can be demonstrated by letting $\alpha_1 = \alpha_2 = 1/2$, $\alpha_3 = \alpha_4 = \alpha_5 = 0$. Then we would have

$$(\frac{T_3^{-1}s^3}{r^3} - w^3(k))^\top T(\alpha) \sum_{h=1}^{5} (\frac{T_h^{-1}s^h}{r^h} - w^h(k)) < 0.$$

5. A DYNAMIC EXTENSION

In this section we will briefly discuss a dynamic extension of the two-period financial model presented in Section 2. The notation from Section 2 is maintained. In this dynamic setting, time is discrete and starts at date 0 and continues forever. Let $N = \{0, 1, 2, \cdots\}$ denote the set of dates. There are n long-lived risky assets, and m long-lived investors each having a stochastic stream of endowments. Trading takes place on each date. While trading takes place, investors will gradually acquire more bits of information over time and therefore adjust their expectations over assets returns and update their perceived mean vectors $s^i \in \mathbb{R}^n$, $i \in I_m$, and their perceived covariance matrices $T_i \in \mathbb{R}^{n\times n}$, $i \in I_m$. So, s^i and T_i are now the functions of time and investor. That is, s^i and T_i will differ across investors and over time. From now on, we will rewrite s^i and T_i as $s^i(t)$ and $T_i(t)$. The matrix $T_i(t)$ is assumed to be symmetric positive definite for all i and t.

On date t, investor i is associated with a utility function $u_t^i : \mathbb{R}^n \mapsto \mathbb{R}$ and an exogenously given endowment $w^i(t) \in \mathbb{R}^n$ and his market portfolio $x^i(t-1)$ transferred from date $t-1$. Let $\omega(t) = \sum_{i=1}^m (w^i(t) + x^i(t-1))$ denote the *market initial endowment* at date t, where $x^i(-1) = 0$ for all $i \in I_m$. It is assumed that $\omega(t) \neq 0$. The utility function at date t has a standard form: $u_t^i(x) = u_t^i(\rho, \sigma) = u_t^i(x^\top s^i(t), \frac{1}{2}x^\top T_i(t)x)$. As in Section 2, u_t^i is assumed to be a strictly concave C^1 function of x with

$$\frac{\partial u_t^i(\rho,\sigma)}{\partial \rho} > 0, \quad \frac{\partial u_t^i(\rho,\sigma)}{\partial \sigma} < 0.$$

Thus, u_t^i is a mean-variance utility function and is an increasing function of the expected return ρ and a decreasing function of the expected variance σ. Similarly, we can define the *risk aversion ratio function* $r_t^i : \mathbb{R}^n \mapsto \mathbb{R}$ at date t as

$$r_t^i(x) = -\frac{\frac{\partial u_t^i(\rho,\sigma)}{\partial \sigma}}{\frac{\partial u_t^i(\rho,\sigma)}{\partial \rho}}.$$

Clearly, r_t^i is a continuous function with positive values. At date t, investor i has a unique satiation portfolio $\bar{x}^i(t)$. Let $\bar{r}_t^i = r_t^i(\bar{x}^i(t))$. Then, we have that $\bar{x}^i(t) = \frac{T_i^{-1}(t)s^i(t)}{\bar{r}_t^i}$. Now let the set $D^i(t) = \{x \in \mathbb{R}^n \mid u_t^i(x) \geq u_t^i(w^i(t)+x^i(t-1))\}$. Clearly, $D^i(t)$ is a nonempty, convex and compact set. Let $V^i(t)$ be the set $\{r_t^i(x) \mid x \in D^i(t)\}$ and let $V(t)$ be the product of all $V^i(t)$, $i \in I_m$. Clearly, $V(t)$ is also a nonempty, convex and compact. For each $\beta = (\beta^1, \cdots, \beta^m) \in V(t)$, we denote $[\frac{T_i^{-1}(t)s^i(t)}{\beta^i} - (w^i(t)+x^i(t-1))]$ by $v_t^i(\beta^i)$ for each $i \in I_m$ and $\sum_{i\in I_m}[\frac{T_i^{-1}(t)s^i(t)}{\beta^i} - (w^i(t)+x^i(t-1))]$ by $v_t(\beta)$. $v_t^i(\beta^i)$ and $v_t(\beta)$ are called *generalized initial endowment of investor* i and *generalized market endowment at date* t, respectively. The total utility function of investor i is an additively separable function over time

$$\sum_{t=0}^{\infty} \rho_i^t u_t^i(x(t))$$

where $x(t) \in \mathbb{R}^n$, $t \in N$, and ρ_i is a discount factor with $0 < \rho_i < 1$.

With respect to this dynamic economy, we will introduce the following concept of dynamic equilibrium.

DEFINITION 5.1. A family $\{p^*(t), t \in N\}$ of vectors with $p^*(t) \in \mathbb{R}^n \setminus \{0\}$ for $t \in N$, is a family of dynamic equilibrium price vectors if there exist $x^{1*}(t)$, $x^{2*}(t)$, $\cdots$, $x^{m*}(t) \in \mathbb{R}^n$ for all $t \in N$ such that

(a) for each $i \in I_m$, $\{x^{i*}(t), t \in N\}$ is a solution of the problem

$$\text{maximize } \sum_{t=0}^{\infty} \rho_i^t u_t^i(x^i(t))$$

$$\text{subject to } \; p^{*\top}(t)x^i(t) \leq p^{*\top}(t)(w^i(t) + x^{i*}(t-1)), \text{ for all } \; t \in N$$

with the convention $x^{i*}(-1) = 0$;

(b) $\sum_{i\in I_m} x^{i*}(t) = \sum_{i\in I_m}(w^i(t) + x^{i*}(t-1))$, for all $t \in N$.

In the definition the family of vectors $x^{1*}(t)$, $\cdots$, $x^{m*}(t)$, $t \in N$, are called *dynamic equilibrium allocations*. Thus, at a dynamic equilibrium, each investor maximizes his total utility under his budget constraint and markets clear for all trading dates.

In the rest of this section, we will discuss several plausible assumptions for this dynamic model.

ASSUMPTION 2. *There exist a vector* $s^* \in \mathbb{R}^n$ *and an* n *by* n *symmetric positive definite matrix* T^* *such that for every investor* $i \in I_m$, *it* *holds*

$$\lim_{t\to\infty} s^i(t) = s^* \text{ and } \lim_{t\to\infty} T_i(t) = T^*.$$

This assumption states that while trading takes place, each investor will adjust his expectations on assets returns, and that as long as investors communicate with and learn from each other and acquire information from markets long enough, their probability beliefs on assets returns will become sufficiently similar and eventually converge to a common probability belief.

For every $\alpha \in S^m$ and every date $t \in N$, let

$$T(t,\alpha) = (\sum_{i\in I_m} \alpha_i T_i^{-1}(t))^{-1}.$$

Such a $T(t,\alpha)$ is called a *market risk-measure matrix* with respect to α and t. Since each $T_i(t)$ is symmetric positive definite, it follows that $T(t,\alpha)$ is also symmetric positive definite.

ASSUMPTION 3. *For every $\alpha \in S^m$, every $t \in N$, every $i \in I_m$ and every $\beta \in V(t)$, it holds that*

$$v_t(\beta) \neq 0 \text{ and } v_t^{i\top}(\beta^i) T(t,\alpha) v_t(\beta) \geq 0.$$

This assumption says that Assumption 1 holds for the dynamic economy on each trading date.

ASSUMPTION 4. *For each investor $i \in I_m$, the total utility function $\sum_{t=0}^{\infty} \rho_i^t u_t^i(x(t))$ is bounded from above.*

This assumption is a simple technical condition closely related to those made for overlapping generations models; see Blanchard and Fisher (1989).

The analysis of this dynamic model will become considerably more difficult than the one for the two-period model in the previous sections and is beyond the scope of the current paper and is thus left to a forthcoming paper.

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