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Portfolio Selection under Parameter Uncertainty using a Predictive Distribution

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We propose a portfolio selection model based on a generalized hyperbolic predictive distribution. This distribution incorporates uncertainties in mean and volatility of market returns. We then select an optimal portfolio with expected utility calculated under the predictive distribution. We demonstrate the performance of the new approach by applying it to simulated and real market data.

Key Words: Portfolio Selection; Parameter Uncertainty; Estimation Error; Bayesian Framework; Predictive Distribution; Generalized Hyperbolic Distribution; Utility Function; Utility Restoration Ratio.
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1. INTRODUCTION

Classical investment decision models assume that the decision maker is given the complete probabilistic model of future returns including true

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parameter values. It is a common practice to estimate these values from exogenous sources such as historical data and to replace true parameter values by them. Thus it is natural that the performance of the decision is affected by the amount of estimation error. It has been known in statistical literature that, when the objective function in the decision is linear or symmetric, loss due to estimation error tends to be small if the estimates are unbiased. However, common utility functions in investment decision are highly non-linear and asymmetric, and moderate estimation error may lead to considerably suboptimal consequences.

The issue of parameter uncertainty in portfolio selection was first investigated by Bawa, Brown and Klein (1979). They discussed the effect of estimation risk on portfolio choice. Using Bayesian framework, Kandel and Stambaugh (1996) pointed out the importance of recognizing the uncertainty. Barberis (2000) examined how the predictability in asset returns affects optimal portfolio choice using the Bayesian framework to model the uncertainty in parameters and to calculate optimal allocation by a simulation method. Jorion (1986), Xia (2001), Kan and Zhou (2004), Garlappi, Uppal and Wang (2005) used the Bayesian predictive approach to account for estimation risk.

We propose another Bayesian predictive approach. We model both mean and volatility of market returns simultaneously, and synthesize a generalized hyperbolic predictive distribution. We then select an optimal portfolio with expected utility calculated under the predictive distribution. We demonstrate the new approach by applying it to simulated and real market data, and assess its performance in the conventional criterion of the expected utility.

2. PREDICTIVE UTILITY

We assume that we have an initial wealth of 1 and the market consists in two investment opportunities; a riskfree investment with rate r and a risky investment with return that depends on a normal random variable X with mean μ and variance $\sigma^2 > 0$. Let $W = W(X)$ be the terminal wealth of a portfolio with w in risky investment and $(1 - w)$ in riskfree investment.

Our risk preference is characterized by a utility function $u(W)$ on the terminal wealth W. If μ and σ^2 are given, then one would maximize the expected utility

$$U(w) = Eu[W(X)] = \int u(W(x)) \cdot d_N(x; \mu, \sigma^2)\, dx \tag{1}$$

where $d_N(x; \mu, \sigma^2)$ denotes normal density with mean μ and variance σ^2.

Suppose instead that a predictive distribution of X is given in a generalized hyperbolic distribution $GH(\lambda, \eta, \beta, \delta, \gamma)$. See Appendix for its derivation and density function d_{GH}. The parameter λ influences kurtosis and η is a location parameter. Linear dependency of excess return on volatility, a common feature in financial markets, is represented by β, which is expressed in the predictive distribution as skewness. δ is a scale parameter and γ determines the shape.

This particular distribution is chosen for its flexibility and ease of update. The class of generalized hyperbolic distributions includes hyperbolic and normal inverse gaussian distributions as special cases. Normal distribution, t-distribution, variance-gamma distributions and many others are obtained as limiting cases. When auxiliary information such as some historical data is given, it is a simple matter of updating parameter values in generalized hyperbolic distribution.

Instead of optimizing (1) with some estimates plugged in, we find a maximum of the predictive utility

$$\tilde{U}(w) = \int u(W(x)) \cdot d_{GH}(x; \lambda, \eta, \beta, \delta, \gamma)\, dx. \tag{2}$$

Numerical solutions of (2) can be obtained by a search method since generalized hyperbolic distributions have finite moments of all orders.

3. NUMERICAL STUDIES

When true parameter values are unknown, we cannot hope to have a portfolio selection rule that is universally optimal for all parameter values. Thus we restrict our attention only to a practical range of parameter values. We consider a hypothetical market where

$$W(X) = (1 - w)e^r + we^{r+X}, \quad X \sim N(\mu, \sigma^2). \tag{3}$$

We choose $r = 0.00061$, $\mu = 0.00018$ and $\sigma = 0.02345$. These values are taken from analyzing weekly Dow-Jones index from 2001 to 2005; annual riskfree rate is about 3% and the mean excess return is about 1%. We adopt the power utility $u(W) = W^{1-A}/(1 - A)$ for $A > 1$ and $\log(W)$ for $A = 1$. We take $A = 1, 2$ for lightly risk-averse investor, $A = 5$ for moderately risk-averse one, and $A = 10$ for extremely conservative one.

We specify prior for (μ, σ^2) as given in Figure 1. It is a moderately diffuse prior; 95% interval for σ^2 ranging from about half of to twice of true σ^2. We experimented with other priors and obtained similar results.

We first generate 12 independent excess returns from $N(\mu, \sigma^2)$ distribution, and determine posterior distribution as outlined in Appendix. Corresponding generalized hyperbolic predictive distribution is used in evalu-

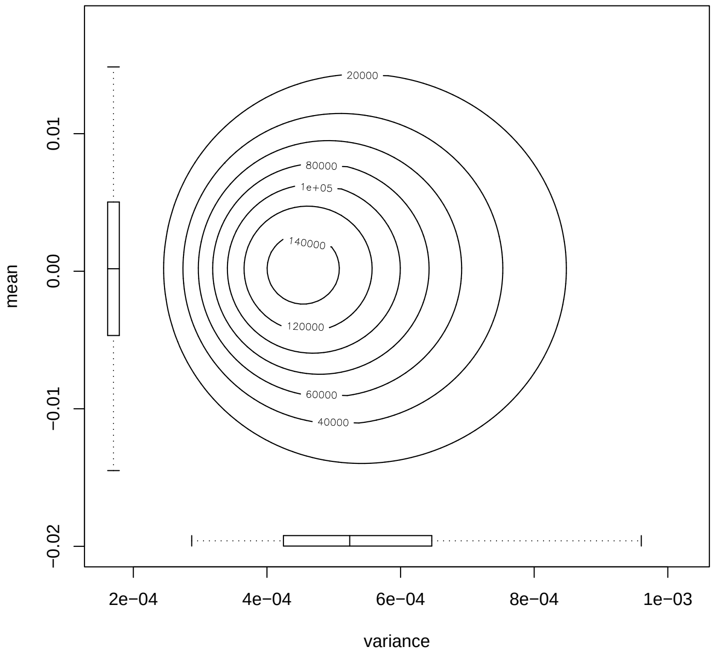

FIG. 1. Normal-GIG prior with $(\eta, \beta, k) = (0.00018, 0, 10)$ and $(\lambda, \delta, \gamma) = (-0.5, \sqrt{10}/0.02345, 0.02345\sqrt{10}, 10)$. Ticks and boxes on the margins denote 2.5, 25, 50, 75, and 97.5 percentiles.

ating (2). An optimal weight w_B is then found by a numerical search. An optimal plug-in choice w_P is also obtained numerically, treating the sample mean and variance as true values. We restrict both w_B and w_P within 0 and 1.

We then generate an independent normal random variable, representing a future return. Terminal wealth is calculated by (3), and utility $u(W)$ is obtained. We repeat the whole procedure 100,000 times, take averages of empirical utilities, and summarized the result in Table 1.

In the table, U^* is the optimal utility obtained with true parameter values known. It is listed for comparison. Columns under U_B and U_P show the average utilities of predictive choice and plug-in choice, respectively. The last column is the *utility restoration ratio*

$$URR = \frac{U_B - U_P}{U^* - U_P},$$

TABLE 1.

Mean ± standard error of simulated utilities.

A	U^*	U_B	U_P	URR
1	4.0874×10^{-4} $\pm 9.5603 \times 10^{-5}$	2.8270×10^{-4} $\pm 9.4243 \times 10^{-5}$	2.8143×10^{-4} $\pm 9.0678 \times 10^{-5}$	1%
2	-0.99943 $\pm 3.8008 \times 10^{-5}$	-0.99991 $\pm 7.9115 \times 10^{-5}$	-1.0001 $\pm 8.8737 \times 10^{-5}$	28%
5	-0.24940 $\pm 1.3319 \times 10^{-5}$	-0.24969 $\pm 3.7378 \times 10^{-5}$	-0.25099 $\pm 8.4126 \times 10^{-5}$	82%
10	-0.11051 $\pm 6.3647 \times 10^{-6}$	-0.11061 $\pm 1.5924 \times 10^{-5}$	-0.11300 $\pm 7.9528 \times 10^{-5}$	96%

measuring the proportion of utility recovered by the predictive method.

TABLE 2.

Basic statistics of weekly excess returns

	mean	volatility	ratio
Dow-Jones index	-3.5986×10^{-4}	2.3467×10^{-2}	-0.015
Microsoft	9.3328×10^{-4}	4.1011×10^{-2}	0.022
Bank of America	2.5763×10^{-3}	3.2022×10^{-2}	0.080
WalMart	-5.8887×10^{-4}	3.2089×10^{-2}	-0.017

We can easily see the superiority of the predictive choice to the plug-in choice, especially when risk-aversion is high. The parameter uncertainty is one of risk factors in investment decision. When the degree of risk-aversion is low, improvement by considering extra risk is marginal. But, when risk-aversion is high, the improvement becomes significant. In an extreme case of $A = 10$, the predictive choice recovers about 96% of utility lost by the simple plug-in selection.

We now apply these selection methods to real market data and evaluate their empirical performances. We take weekly returns of Dow-Jones index, MicroSoft, Bank of America and WalMart stocks from July 5, 2001 to December 30, 2005. The 3-years Treasury bills is used for risk-free investment, which shows average weekly return of 6.1363×10^{-4} with volatility 1.7688×10^{-4}. We obtain optimal allocations and utility values using the same procedure as described earlier. For each week, we take data from the most recent 12 weeks as historical data. However, we cannot calculate URR because we don't have the true parameter values.

The results as summarized in Table 3 show a remarkable similarity to the ones from the hypothetical data. Because of relatively small sample size, we do not list sample standard errors. Again, the table suggests the

TABLE 3.

Empirical utilities from real market data

	A	U_B	U_P
Dow-Jones Index	1	5.6934×10^{-4}	2.6649×10^{-4}
	2	-9.9996×10^{-1}	-9.9998×10^{-1}
	5	-2.4948×10^{-1}	-2.5000×10^{-1}
	10	-1.1055×10^{-1}	-1.1112×10^{-1}
MicroSoft	1	-9.5633×10^{-4}	-6.5526×10^{-4}
	2	-1.0010×10^{0}	-1.0008×10^{0}
	5	-2.5027×10^{-1}	-2.5174×10^{-1}
	10	-1.1094×10^{-1}	-1.1309×10^{-1}
Bank of America	1	9.0640×10^{-4}	1.9894×10^{-4}
	2	-9.9948×10^{-1}	-1.0000×10^{0}
	5	-2.4948×10^{-1}	-2.5028×10^{-1}
	10	-1.1055×10^{-1}	-1.1154×10^{-1}
WalMart	1	-7.5457×10^{-4}	-7.7668×10^{-4}
	2	-1.0006×10^{0}	-1.0007×10^{0}
	5	-2.4996×10^{-1}	-2.5086×10^{-1}
	10	-1.1079×10^{-1}	-1.1225×10^{-1}

superiority of the predictive approach to the plug-in method, except one or two cases when risk-aversion is low.

4. CONCLUSION

Models for financial decision often involve unknown parameters which must be obtained from somewhere else with estimation error. Selecting a portfolio under this additional risk is thus subject to some loss of utility.

As an alternative to conventional plug-in method of choosing the most probable predictive distribution, we proposed a Bayesian predictive distribution which is an average of plausible distributions. It is a natural framework of taking into account of the additional uncertainty caused by estimation. We demonstrated the performance of the predictive approach using some synthesized and real data. The improvement was especially significant when the degree of risk-aversion is high.

However, as mentioned earlier, there is no portfolio selection rule that is uniformly optimal for all cases when there is uncertainty in parameter values. When we make an investment decision, we have to analyze the market thoroughly and assess our own risk preferences carefully before choosing a portfolio selection rule. If we feel that we are risk-averse and the predictability of future return is in doubt, we recommend to consider the proposed approach in decision making.

APPENDIX: GENERALIZED HYPERBOLIC PREDICTIVE DISTRIBUTIONS

Let X be a normal random variable with mean μ and variance σ^2. Assume that we have a prior distribution for (μ, σ^2) specified by

$$\begin{aligned} \mu|\sigma^2 &\sim N(\eta_0 + \beta_0\sigma^2, \sigma^2/k_0), \\ \sigma^2 &\sim GIG(\lambda_0, \delta_0, \gamma_0), \end{aligned}$$

for some η_0, β_0, $k_0 > 0$, λ_0, $\delta_0 > 0$, and $\gamma_0 > 0$. Here, $GIG(\lambda, \delta, \gamma)$ stands for a generalized inverse gaussian distribution with density

$$d_{GIG}(z; \lambda, \delta, \gamma) = \frac{\gamma^\lambda}{2\delta^\lambda K_\lambda(\delta\gamma)}\, z^{\lambda-1} e^{-\frac{1}{2}(\delta^2 z^{-1} + \gamma^2 z)}, \quad z > 0,$$

for $-\infty < \lambda < \infty$, $\delta > 0$, $\gamma > 0$, where K_λ is the modified Bessel function of the third kind with index λ. The GIG class is highly flexible and has been suggested as a plausible volatility model by several authors including Barndorff-Nielsen and Shephard (2001). Especially, when $\lambda = -1/2$, it becomes an inverse Gaussian distribution with mean δ/γ and variance δ/γ^3. Hence, if our prior knowledge is such that (μ, σ^2) seems to be centered around $(\hat{\mu}, \hat{\sigma}^2)$, but with high degrees of uncertainty, we may choose k_0, δ_0, γ_0 small with $\delta_0/\gamma_0 = \hat{\sigma}^2$ so that the prior distribution is almost flat.

Prior distribution (**??**) can be easily updated with additional information. Let $\underline{\mathbf{x}} = (x_1, x_2, \ldots, x_n)$ be an observed data from a normal distribution with mean μ and variance σ^2. Then a tedious but straightforward calculation gives the posterior distribution

$$\begin{aligned} \mu|\sigma^2 &\sim N(\eta + \beta\sigma^2, \sigma^2/k), \\ \sigma^2 &\sim GIG(\lambda, \delta, \gamma), \end{aligned}$$

where $k = k_0 + n$, $\eta = (k_0\eta_0 + n\overline{x})/k$, $\beta = k_0\beta_0/k$, $\lambda = \lambda_0 - n/2$,

$$\delta = \left\{\delta_0^2 + \sum_{i=1}^{n}(x_i - \overline{x})^2 + \frac{k_0}{k}n(\overline{x} - \eta_0)^2\right\}^{1/2}, \text{ and } \gamma = \left\{\gamma_0^2 + \frac{k_0}{k}n\beta_0^2\right\}^{1/2}.$$

With this posterior distribution, the predictive distribution of X is easily obtained since it is a well-known mean-variance mixture of normal distribution, called a generalized hyperbolic (GH) distribution with parameters

λ, η, β, $\tilde{\delta} = \delta\sqrt{(k+1)/k}$, and $\tilde{\gamma} = \gamma\sqrt{k/(k+1)}$. See Eberlein and Hammerstein(2004) and Barndorff-Nielsen and Stelzer(2004) for more detailed discussions on this distribution. Its density is given by

$$d_{GH}(x;\lambda,\eta,\beta,\tilde{\delta},\tilde{\gamma}) = \frac{\tilde{\gamma}^{\lambda}}{\sqrt{2\pi}\alpha^{\lambda-\frac{1}{2}}K_{\lambda}(\tilde{\delta}\tilde{\gamma})}d(x)^{\lambda-\frac{1}{2}}e^{\beta(x-\eta)}K_{\lambda-\frac{1}{2}}(\alpha d(x)),$$

where $\alpha = \sqrt{\tilde{\gamma}^2+\beta^2}$ and $d(x) = \sqrt{\tilde{\delta}^2+(x-\eta)^2}$.

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