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Power Risk Aversion Utility Functions

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This paper introduces a new class of utility functions—the power risk aversion. It is shown that the CRRA and CARA utility functions are both in this class. The implications of the PRA utility functions are explored in the context of growth theory. In particular, it is found that economies facing a common real interest rate do not necessarily grow at the same rates if they start with different levels of capital stock. Thus diversity in growth performance across countries occurs even if these countries have access to perfect international capital markets. Potential applications of the PRA in asset pricing are considered.

Key Words: PRA utility functions; Growth; Asset pricing.
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1. INTRODUCTION

Different forms of utility functions are used in the economics profession, some chosen according to the ease of theoretical exposition, others according to the need for model calibrations and simulations. By far the most widely used utility functions are the ones with constant relative risk aversion (CRRA). In the literature of new growth theory alone for instance, Lucas (1988), Romer (1990), and Grossman and Helpman (1991) demonstrate its application.

Although there is no decisive empirical evidence for the truth of constant relative risk aversion, its use in dynamic macroeconomic models is popular due to its analytical ease. The danger of this practice lies in the propensity to take the special results as general conclusions. In a model with a long

* The views expressed in this paper are those of the author and do not necessarily represent those of the IMF or IMF policy.

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lived representative agent for example, if the momentary utility function is CRRA with parameter σ, and if the rate of time preference is ρ, then economies facing a common real interest rate r should grow at the same rate $(r-\rho)/\sigma$. If we are not careful, we may conclude that the observed diversity in growth rates across countries under relatively complete capital markets represents a puzzle. To prevent such sloppy thinking, access to a broader class, escaping the narrow focus of the CRRA utility functions, is essential. The hyperbolic absolute risk aversion (HARA) family of utility functions, which was introduced in Merton (1971) and has subsequently gained popularity in finance literature, seems to be a poor candidate for adoption in macroeconomic studies, especially when economic growth is of interest, because half of the members in the HARA family are simply not well defined for high levels of consumption.

Other utility functions are sporadically tried. For example, a modified version of the CARA is used in Frank (1990) and the Stone-Geary type is used in Rebelo (1992). These utility functions, due to their inconvenience or sometimes due to their lack of serious promotion, have not been widely adopted in macroeconomic studies.

This paper offers a new class of utility functions which addresses these problems and inadequacies. The new class, which we call the *power risk aversion*, is defined and described in Section 2. We show that the PRA contains the CRRA and CARA as special examples as the HARA does. In Section 3, we study the implications of the PRA utility functions in the context of growth theory. We find that with the new utility functions, the diversity in growth rates across countries can arise even if these countries have access to complete international capital markets. Rebelo (1992) has reached the same conclusion. The difference between his work and our paper is important enough to warrant a separate treatment in Section 5 after Section 4 presents a long-run comparative dynamics analysis. Section 6 provides concluding comments, speculating on the use of the PRA utility functions in resolving the equity premium puzzle.

2. A NEW CLASS OF UTILITY FUNCTIONS

The new class of utility functions — the power risk aversion — is defined as follows:

$$u(c)=\frac{1}{\gamma}\left[1-\exp\left(-\gamma\left(\frac{c^{1-\sigma}-1}{1-\sigma}\right)\right)\right],\ \ \sigma\geq 0,\ \gamma\geq 0. \tag{1}$$

When $\gamma=0$, $u(c)$ is defined as the limit of right-hand side as γ approaches zero. The same applies to the case when $\sigma=1$.

Proposition 1. *The PRA utility functions defined in (1) have the following properties:*

1. $u' > 0$ *and* $u'' < 0$*. Thus,* $u(\cdot)$ *is indeed increasing and concave.*
2. When $\gamma = 0$*,* $u(c) = \left(c^{1-\sigma} - 1\right) / (1 - \sigma)$ *and is thus CRRA.*
3. When $\sigma = 0$*,* $u(c) = \left(1 - e^{-\gamma(c-1)}\right) / \gamma$ *and is thus CARA.*
4. $RRA = \sigma + \gamma c^{1-\sigma}$*;* $ARA = \sigma c^{-1} + \gamma c^{-\sigma}$*.*
5. When $\gamma > 0$*,* $u(c)$ *is bounded from above.*

Proof. To prove property 2, use L'Hospital's Rule. The proofs of other properties are straightforward and hence omitted. ▮

Remark 2.1. Property 4 justifies the name PRA given to the new class of utility functions: the two risk aversion measures (relative and absolute) are both power functions of consumption. Whereas the coefficient of absolute risk aversion is always decreasing or constant in the level of consumption, the coefficient of relative risk aversion can be decreasing (when $\sigma > 1$), increasing (when $\sigma < 1$) or constant (when $\sigma = 1$). Property 5 implies that with positive γ, the existence problem disappears in dynamic models with time discounting no matter what value σ takes.

Proposition 2. *If* $\gamma_1 < \gamma_2$*, then*

$$u(c; \sigma, \gamma_1) \geq u(c; \sigma, \gamma_2) \text{ for any } c, \text{ with equality iff } c = 1.$$

Proof. For fixed c and σ, we calculate the derivative of u with respect to parameter γ.

$$\begin{aligned} \frac{\partial u}{\partial \gamma} &= \frac{1}{\gamma^2}\left\{\left[\gamma\left(\frac{c^{1-\sigma}-1}{1-\sigma}\right)+1\right]\exp\left(-\gamma\left(\frac{c^{1-\sigma}-1}{1-\sigma}\right)\right)-1\right\} \\ &= \frac{1}{\gamma^2} g(x), \\ \text{where } g(x) &= (x+1)\mathrm{e}^{-x} - 1, \\ \text{with } x &= \gamma\left(\frac{c^{1-\sigma}-1}{1-\sigma}\right). \end{aligned}$$

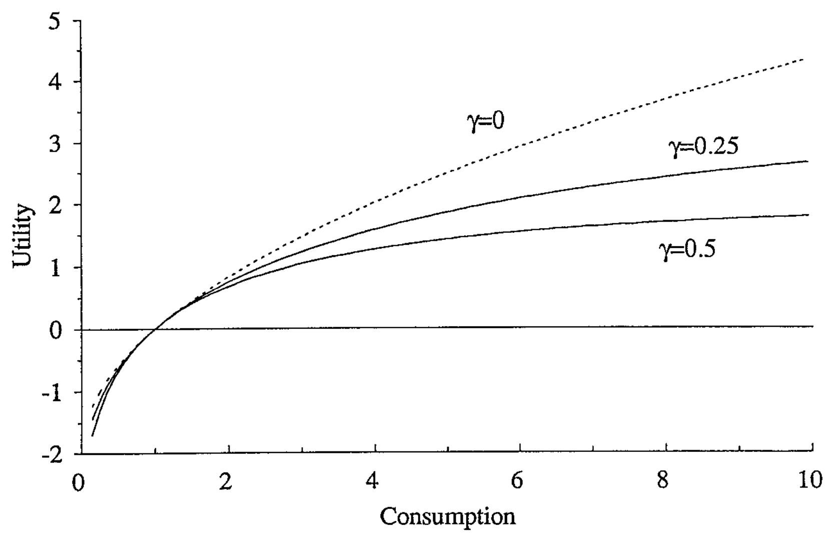
Next, we show that $g(x) \leq 0$ for any x and with equality iff $x = 0$. To establish that, we calculate $g'(x)$ and find that,

$$g'(x) = -x\mathrm{e}^{-x} = \begin{cases} + & \text{if } x < 0 \\ 0 & \text{if } x = 0 \\ - & \text{if } x > 0 \end{cases}$$

Thus $g(x)$ attains a *strict* maximum at $x = 0$. Note that $g(0) = 0$ and that $x = 0$ iff $c = 1$. This ends our proof. ▌

Remark 2.2. Proposition 2 indicates that the higher the γ, the more curved the utility function. See Figure 1 for a graphical description of the proposition.

FIG. 1. How Utility Function Changes with γ



PROPOSITION 3. *If* $\sigma_1 < \sigma_2$, *then*

$$u(c;\sigma_1,\gamma) \geq u(c;\sigma_2,\gamma) \text{ for any } c, \text{ with equality iff } c = 1.$$

Proof. For fixed c and γ, we calculate the derivative of u with respect to parameter σ.

$$\begin{aligned} \frac{\partial u}{\partial \sigma} &= \left\{ \frac{c^{1-\sigma}[1-(1-\sigma)\ln(c)]-1}{(1-\sigma)^2} \right\} \exp\left(-\gamma\left(\frac{c^{1-\sigma}-1}{1-\sigma}\right)\right), \\ &= \frac{h(c)}{(1-\sigma)^2} \exp\left(-\gamma\left(\frac{c^{1-\sigma}-1}{1-\sigma}\right)\right), \\ \text{where} \quad h(c) &= c^{1-\sigma}\left[1-(1-\sigma)\ln(c)\right]-1. \end{aligned}$$

It remains to show that $h(c) \leq 0$ for any $c \in [0, +\infty)$ and with equality iff $c = 1$. To establish that, we calculate $h'(c)$ and find that,

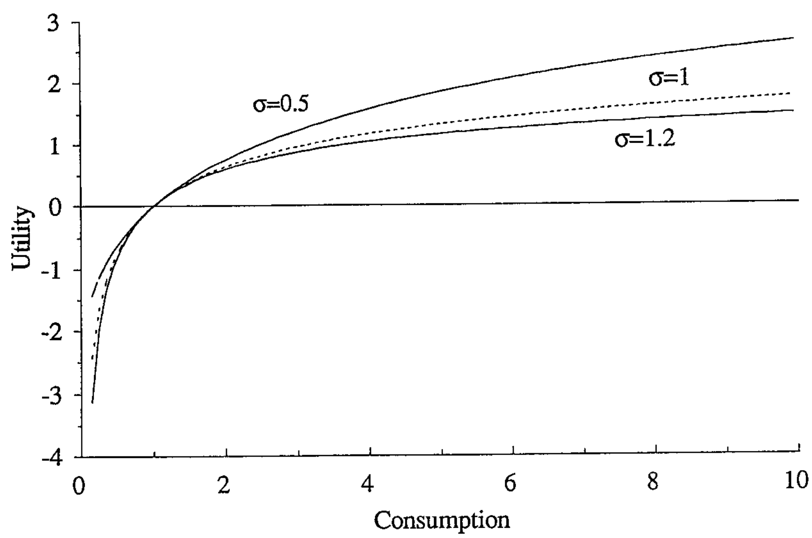
$$h'(c) = -(1-\sigma)^2 c^{-\sigma} \ln(c) = \begin{cases} + & \text{if } c < 1 \\ 0 & \text{if } c = 1 \\ - & \text{if } c > 1 \end{cases}$$

Thus $h(c)$ attains a *strict* maximum at $c = 1$. As a result, $h(c) < h(1) = 0$ if $0 \leq c \neq 1$. ▮

Remark 2.3. Proposition 3 indicates that the higher the σ, the more curved the utility function. See Figure 2 for a graphical description of the proposition.

Although both a greater γ and a greater σ make the utility function more curved, there is a major difference in their effect on risk aversion. A greater γ always increases risk aversion; a greater σ on the other hand, twists the coefficients of risk aversion (when $\gamma \neq 0$), raising them for one range of consumption and lowering them for the other range (see property 4 in Proposition 1).

FIG. 2. How Utility Function Changes with σ



3. ECONOMIC IMPLICATIONS

We focus on the implications of the PRA utility functions in the literature of growth theory.

One difficulty in growth theory is the necessity to account for the diversity of growth rates across countries. A special collection of papers in JPE 1990 partly deals with this problem.

Lucas (1990) and King and Rebelo (1993) concluded that neoclassical growth models without exogenous technological change cannot explain what is observed about growth rates if countries are assumed to have access to the same technology. Adding exogenous technological change to the neoclassical growth models is not to solve the problem but to finesse it.

Lucas (1988) and Romer (1990) show that countries can grow at different speeds if they start with different amounts of physical and/or human capital stocks. Benhabib and Perli (1994) and Xie (1994) argue that the diversity in growth performance across countries may be due to the existence of multiple equilibria. In all these works among others, the explanations for the diversity rely on the diversity in the equilibrium real interest rates across countries. The models are for closed economies. Once international trade is introduced into these models, the diversity will disappear when the differences in the real interest rates are removed. This is because with the CRRA utility function assumed in these models, the equalization of the real interest rates leads to the equalization of the growth rates since the rate of growth is simply $(r-\rho)/\sigma$, where r is the common real interest rate, σ is the coefficient of relative risk aversion, and ρ is the rate of time preference.

Once we extend the preferences to the power risk aversion utility functions, we can explain the different growth rates across countries even if international trade gives all the countries access to a common technology, as long as these countries start with different levels of capital stock. To fix ideas, let the common technology be represented by a production function such as Ak, whose use has been popularized by Rebelo (1991) and Barro (1990).

Consider an economy with long-lived representative agents whose preferences are given by,

$$\int_0^\infty u(c)e^{-\rho t}\,dt \tag{2}$$

where $u(c)$ is a PRA utility function defined by equation (1). Thus the equilibrium allocation can be obtained by solving the maximization problem of a social planner who maximizes (2), subject to:

$$\dot{k} = Ak - c, \text{ and } k(0) \text{ given.} \tag{3}$$

The Hamiltonian can be written as

$$H = u(c) + \lambda(Ak - c). \tag{4}$$

The first order conditions are given by,

$$c^{-\sigma} \exp\left(-\gamma\left(\frac{c^{1-\sigma}-1}{1-\sigma}\right)\right) = \lambda \tag{5}$$

$$\dot{\lambda} = \lambda(\rho - A) \tag{6}$$

The transversality condition is: $\lambda k e^{-\rho t} \to 0$ as $t \to \infty$.

To allow for economic growth, we assume that $A > \rho$. That is, the productivity of capital is high compared to the rate of time preference. Since A is in fact the equilibrium real interest rate, we are assuming that the common real interest rate facing the economies is greater than ρ.

It is routine to show that

$$\frac{\dot{c}}{c} = \frac{(A-\rho)}{RRA} \tag{7}$$

where RRA is given in Proposition 1. Equation (7) has several implications, which are put in a proposition as follows.

PROPOSITION 4. *If $\gamma = 0$, or if $\sigma = 1$, then economies will grow at the same rate regardless of their starting positions. Otherwise, economies will grow at different rates that depend on the initial level of capital stock.*

Proof. Since $RRA = \sigma + \gamma c^{1-\sigma}$, $\gamma = 0$ or $\sigma = 1$ implies that RRA is constant in c. Thus equation (7) indicates that in these two cases, the growth rate of consumption is constant and is independent of the starting positions of the economies.

When $\gamma > 0$, economies with higher capital stock will grow faster (slower) if σ is greater (less) than unity. This is because with $\gamma > 0$, RRA is decreasing (increasing) in c when σ is greater (less) than unity; and c is positively related to capital stock. ∎

Remark 3.1. Growth rate of *consumption* and economic growth are apparently used interchangeably. This is justified because: First, equilibrium rate of growth of *consumption* is the point of critical concern, and second, consumption and capital stock and thus output will grow at the same rate in the long run (see the following proposition).

PROPOSITION 5. *As t goes to infinity, c/k approaches a positive constant. Thus in the long run, the growth rates of capital and of output are the same as the growth rate of consumption. More specifically, we have*

$$\lim_{t\to\infty} c/k = \begin{cases} A & \text{if } 0 \leq \sigma < 1 \\ A - \frac{A-\rho}{1+\gamma} & \text{if } \sigma = 1 \\ A - \frac{A-\rho}{\sigma} & \text{if } \sigma > 1 \end{cases}$$

Proof. See the appendix. ▮

We have seen from Proposition 4 that economies can grow at different speeds, depending on their stage. Fitting this model to actual observations seems to require $\sigma < 1$ because this will imply that richer countries grow more slowly. A simple numerical example demonstrates how the model generates the growth diversity observed in the world. Before we assign the values to the parameters, let us get some quantitative feeling about how much the diversity in growth rates is really observed.

The sample we consider includes 22 countries classified in the World Development Report (1993) as high income economies. When using this sample, the assumption of a common real interest rate is more valid than when using a larger sample. We derive the 1991 per capita consumption data from the World Development Report (1993) for the sample and then normalize it so that the highest consumption is 1. The series is reported as Column 2 in Table 1. Column 3 in Table 1 gives the average annual growth rate of private consumption for the period 1980–1991. When we plot the growth rates against the level of consumption in Figure 3, we see a rough negative relation: richer countries have lower growth rates of consumption. The following linear regression confirms the visual impression:

$$g_c = 5.57 - \frac{(-2.57)}{4.07}c, \quad R^2 = 0.25,$$

where t-statistics is put in parenthesis.

Now the question is, can we pick the values of the parameters in this model to generate a diagram similar to Figure 3? Indeed we can. Let $\rho = 0.02$, $A = 0.04$, $\sigma = 0.2$, and $\gamma = 1$. We obtain that, for the country with the highest level of consumption (normalized to 1), the growth rate of per capita consumption is 1.7%. The growth rate of per capita consumption for a country with consumption level at 0.5 is 2.6 %, and that for a country with consumption level at 0.3 is 3.4 %. See Table 2 and Figure 4 for details.

In this exercise, we are not trying to find the best fit of the model to the real world observations. We are merely asserting the possibility of a fit using a simple example.

TABLE 1.

Observed Diversity

Countries	Consumption	Growth(%)
Singapore	0.32	6.0
Ireland	0.34	2.2
Hong Kong	0.38	6.9
Spain	0.41	3.2
New Zealand	0.41	1.9
Israel	0.43	5.2
Australia	0.58	3.2
Netherlands	0.58	3.2
United Kingdom	0.59	3.8
Belgium	0.61	1.9
Italy	0.62	3.1
Austria	0.62	2.6
France	0.67	2.4
Canada	0.69	3.4
Germany	0.71	2.2
Norway	0.73	1.2
Japan	0.74	3.7
Denmark	0.76	1.7
United States	0.79	2.4
Finland	0.80	4.3
Sweden	0.85	2.0
Switzerland	1.00	1.7

Consumption: Level of Consumption Relative to that of Switzerland, 1991
Growth: Average Annual Growth Rate of Private Consumption 1980–1991
Depicted in Figure 3

To summarize, this section examines the idea that the diversity in economic growth rates among countries merely reflects the fact that countries are at different economic stages. In other words, countries are located at different positions along a transitional path. Proposition 4 shows that this idea works if the utility function has power risk aversion with parameter $\gamma > 0$ and $\sigma < 1$, i.e. if the utility function displays increasing relative risk aversion.

It was once argued that neoclassical growth models can also explain the diversity in growth rates across countries by placing these countries at different positions along a transitional path. Recent studies by Lucas (1990) and King and Rebelo (1993) point out that these models imply

FIG. 3. Observed Diversity

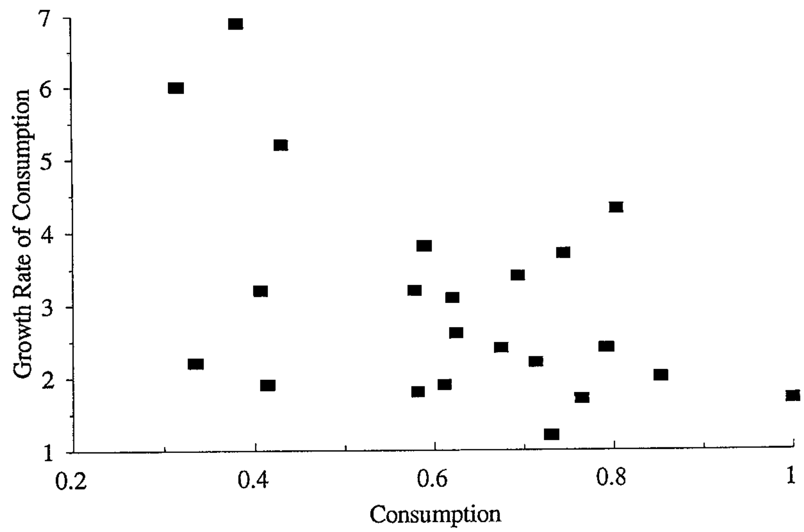


TABLE 2.

Generated Diversity

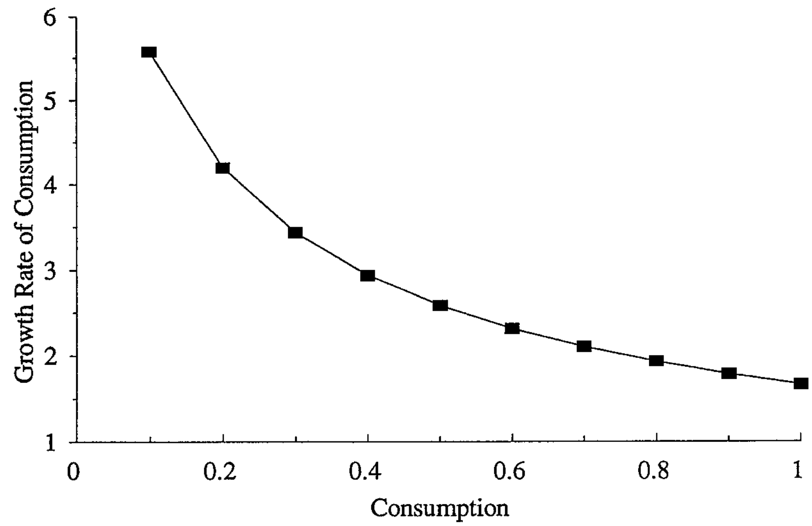
Consumption	Growth(%)
0.10	5.6
0.20	4.2
0.30	3.4
0.40	2.9
0.50	2.6
0.60	2.3
0.70	2.1
0.80	1.9
0.90	1.8
1.00	1.7

Consumption: Level of Consumption Relative to the Highest
Growth: Annual Rate of Growth of Consumption Depicted in Figure 4

differences in the real interest rate across countries that are too large to be consistent with the slow movement of capital across borders.

There is one thing in our analysis that may also be bothersome. In order to fit the model with the reality, we need to impose $\sigma < 1$. This means

FIG. 4. Generated Diversity



that the representative individual has an increasing relative risk aversion. Is this assumption counter-intuitive?

Ever since the risk aversion measures were proposed by Arrow (1970) and Pratt (1964), the consensus view is that the absolute risk aversion should be decreasing in wealth. As to the *relative* risk aversion, no such consensus has emerged. Frank (1990) speculates that diminishing relative risk aversion is more reasonable. Pratt (1964) suggests that the relative risk aversion should be first decreasing and then increasing. Empirical work is therefore needed to determine whether individuals with increasing relative risk aversion represent the majority.

4. LONG-RUN ANALYSIS

In this section, we would like to understand how the parameters in the utility function affect growth in the long run.

PROPOSITION 6. *In the non-trivial case of $\gamma > 0$, the long-run growth rate is given by*

$$g_c = g_k = \begin{cases} 0 & \text{if } 0 \leq \sigma < 1 \\ \frac{A-\rho}{1+\gamma} & \text{if } \sigma = 1 \\ \frac{A-\rho}{\sigma} & \text{if } \sigma > 1 \end{cases}$$

Proof. Note that $g_k = A - c/k$. Proposition 6 is a direct corollary of Proposition 5. ▮

Remark 4.1. In the appendix, it is shown that consumption and capital both increase without bound. With that in mind, it is understandable that parameter γ has no effect on the long-run growth rate if $\sigma \neq 1$, because when $\sigma \neq 1$, γ only affects the RRA in the short run. In the long run, since consumption approaches infinity, RRA either approaches σ (when $\sigma > 1$) or ∞ (when $\sigma < 1$), and is independent of γ in both cases. This result may lead to misinterpretation that introducing parameter γ into the utility function adds practically nothing to the long-run analysis. In fact, it contributes a great deal for two reasons.

First, the long-run behavior is very much different in the case when $\gamma > 0$ and when $\gamma = 0$. In the latter case, the long-run rate of growth is always $(A - \rho)/\sigma$ regardless of what value σ takes on. Proposition 6 shows that when $\gamma > 0$, the long-run rate of growth has a different formula depending on what region σ lies in. It is unfortunate that the literature so far has focused on the narrow and indeed the zero probability case of $\gamma = 0$.

Second, although when $\sigma \neq 1$, the long-run rate of growth is unchanged if γ changes from one positive value to another, the long-run *level* of consumption will change. When $\sigma < 1$, the case we should focus on as we explained in the last section, the growth rate converges to zero. But still the consumption level goes to infinity and we have the following proposition.

PROPOSITION 7. *When $\sigma < 1$, the lower γ is, the higher consumption will be in the long run. More specifically, if $\gamma_1 < \gamma_2$, then*

$$\frac{c_1(t)}{c_2(t)} = \left(\frac{\gamma_2}{\gamma_1}\right)^{1/(1-\sigma)} > 1 \text{ as } t \text{ approaches infinity.}$$

Proof. It is shown in the appendix (see equation (A.6)) that when $\sigma < 1$, we have

$$\lim_{t\to\infty} \frac{\gamma c(t)^{1-\sigma}}{(1-\sigma)(A-\rho)t} = 1, \tag{8}$$

which proves the proposition. ▮

Remark 4.2. Proposition 7 paints an interesting picture. Suppose country 1 and country 2 have the same σ but with $\gamma_1 < \gamma_2$. Also suppose country

1 has lower initial endowment of physical capital. The fact that eventually $c_1(t)$ (and $k_1(t)$) will be greater than $c_2(t)$ (and $k_2(t)$) must mean that country 1 has a higher saving rate initially than country 2 does, making the overtaking possible. Overtaking has been shown possible in the literature when two countries have different rates of time preference. The traditional argument maintains that more patient people (smaller ρ) will accumulate their wealth more rapidly. Here, to tell a similar story, we may interpret parameter γ as the *rate of satiation*. A smaller γ makes the utility function less curved, and the satisfaction approaches a higher level $1/\gamma$ as consumption increases without bound—appropriate characteristics of low rate of satiation. Interpreted this way, Proposition 7 states that a country with a lower rate of satiation shall accumulate greater wealth and experience more consumption in the long run. Note that, throughout the discussion, the assumption $\sigma < 1$ is maintained.

PROPOSITION 8. *For fixed $\gamma > 0$, when $0 \leq \sigma_1 < \sigma_2 < 1$, we have*

$$\lim_{t\to\infty} \frac{c_1(t)}{c_2(t)} = 0$$

Proof. Straightforward from equation (8). ∎

Remark 4.3. Proposition 7 and 8 indicate that whereas a lower γ leads to higher long-run level of consumption, a lower σ apparently has the opposite effect even though they both make the utility function less curved. This reflects the difference in the ways they affect the RRA. We mentioned this difference in Section 2. Whereas a smaller γ lowers the RRA unambiguously, a smaller σ (with a positive γ) serves to twist the RRA and more importantly it raises the RRA in the long run as consumption approaches infinity.

Just for completeness, in the case when $1 < \sigma_1 < \sigma_2$, $c_1(t)$ grows faster than $c_2(t)$ because now a smaller σ lowers the RRA in the long run and speeds up economic growth (see Proposition 6).

5. DIFFERENCE WITH REBELO'S ANALYSIS

The 22 countries in our sample make up only a small fraction of the world. Thus our analysis explains a portion of the whole diversity picture. Rebelo (1992) explains another portion.

The world-wide diversity in economic growth can be described by the following stylized facts, which we draw from Lucas (1988), Barro (1991) and the diagrams presented in Romer (1989).

Fact 1. Low income countries grow more slowly than high income countries.

Fact 2. Mid income countries grow more rapidly than high income countries.

Fact 3. Within high income countries, a richer country grows more slowly.

Rebelo (1992) realizes that the CRRA[1] utility functions have implausible implications on growth equalization under perfect international capital markets. He proposes to use the Stone-Geary type of utility functions to replace the CRRA, and he shows that the diversity in growth performance remains even if the poor countries have access to perfect international capital markets. In particular, the poor countries will still save less and grow more slowly. His thought experiment implies that with policies that improve international capital markets, Fact 1 remains true.

We propose in this paper to replace the CRRA by the PRA utility functions and succeed in explaining Fact 3. The explanation is also consistent with Fact 2 if we accept the assumption of perfect international capital markets among mid and high income economies as a good approximation of the real world.

Whereas Fact 3 and possibly Fact 2 contradict Rebelo's analysis, Fact 1 does not contradict ours because the assumption of perfect international capital markets is violated when low income countries are included in discussion. We believe if all countries have access to a common technology and perfect international capital markets, the poorer countries shall be able to grow faster than the richer ones in the short run and experience the same rate of growth in the long run (see Section 3 for the short-run and Section 4 for the long-run analysis). Since our argument hinges on the assumption $\sigma < 1$, confirmation from empirical work on human behavior is needed to support our belief.

To summarize, this paper and Rebelo (1992) differ in that his use of the Stone-Geary utility function implies *decreasing* relative risk aversion, and our PRA utility function with $\sigma < 1$ implies *increasing* relative risk aversion. As a result, his view is pessimistic: Poor countries will always grow more slowly than the rich ones even if a common technology is shared by all. Our view is optimistic: Access to a common technology shall lead to convergence in growth rate in the long run.

[1] In his paper, Rebelo adopts the name CES utility functions rather than the CRRA. We prefer the latter because the elasticity of substitution is not well defined for some utility functions, for example the Stone-Geary type he uses, unless we make it a convention to define the elasticity of substitution as the inverse of the RRA.

As much as the two papers differ in results, they send a common message that empirical and theoretical work on the preference side can enhance significantly our understanding of economic growth.

6. CONCLUDING COMMENTS

We introduced in this paper a new class of utility functions—the power risk aversion utility functions, which we believe has attractive neatness and interesting properties. It contains the usual CRRA and CARA as special examples. We showed that allowing the relative risk version to change with the level of consumption may be one way to explain the diversity of growth rates across countries. Long-run analysis points to a possible interpretation of parameter γ — the *rate of satiation*. People with a low rate of satiation will eventually end up with high consumption.

Although the application of the PRA utility functions in growth theory is significant, more fruitful applications of it may be in micro studies of human behavior and in asset pricing. Before, when we needed to use a specific form of utility functions, whether for calibration, simulation, or testing, we relied heavily on the CRRA , CARA, or HARA (which include quadratic form). However, no decisive empirical studies have confirmed the validity of the assumption of constant relative or absolute risk aversion. Also, HARA is not easy to manipulate and has limited regions in which it is well defined. The PRA utility functions should prove a better alternative. For example, it may help resolve the equity premium puzzle. Since the PRA utility functions include CRRA as special cases, the use of these utility functions can only enlarge the admissible region in the space of risk-free rate of return and equity premium, depicted in Mehra and Prescott (1985). We speculate that the enlarged admissible region may contain the observed combination of risk-free rate and equity premium. This, however, must remain speculation until confirmed by our fellow economists with advanced computational technique.

APPENDIX

In this appendix, we prove Proposition 5 by a series of lemmas.

LEMMA 1. $\lim_{t\to\infty} \dot{c}/c = \xi < A$.

Proof. The following is not the easiest way to prove the lemma. It is presented here because it gives the implicit solution of the optimal consumption path.

We know that the optimal consumption path is governed by the first order differential equation,

$$\dot{c}/c = \frac{A-\rho}{\sigma+\gamma c^{1-\sigma}}, \tag{A.1}$$

When $\sigma = 1$,we have

$$\dot{c}/c = (A-\rho)/(1+\gamma). \tag{A.2}$$

Thus the lemma is obviously true in this case.

When $\sigma \neq 1$, we do a change of variable, $z = c^{1-\sigma}$. Then we have $\dot{z}/z = (1-\sigma)\dot{c}/c$. The differential equation (A.1) becomes

$$(\sigma+\gamma z)\frac{\dot{z}}{z} = (1-\sigma)(A-\rho). \tag{A.3}$$

The general solution to this equation is given by,

$$\sigma \ln|z| + \gamma z = (1-\sigma)(A-\rho)t + (1-\sigma)J, \tag{A.4}$$

where J is any constant. The optimal consumption path corresponds to a particular J such that the transversality condition is satisfied. For the proof of our lemma, however, there is no need to determine J. Equation (A.4) can be rewritten as

$$\sigma \ln c + \frac{\gamma}{1-\sigma}c^{1-\sigma} = (A-\rho)t + J. \tag{A.5}$$

By assumption, A is greater than ρ. Thus equation (A.5) implies that $c(t)$ approaches ∞. This knowledge helps us determine the long-run growth rate of consumption, which we calculate for two subcases, $0 \leq \sigma < 1$, and $\sigma > 1$.

When $0 \leq \sigma < 1$, the second term in the left-hand side of equation (A.5) is the dominant term as t approaches infinity. Hence we have

$$\lim \frac{\gamma c^{1-\sigma}}{(1-\sigma)(A-\rho)t} = 1. \tag{A.6}$$

When $\sigma > 1$, the first term in the left-hand side is the dominant term as t approaches infinity. Thus, we have

$$\lim \frac{\sigma \ln c}{(A-\rho)} = 1. \tag{A.7}$$

Combining equations (A.2), (A.6), and (A.7), we obtain

$$\lim_{t\to\infty} \dot{c}/c = \begin{cases} 0 & \text{if } 0 \le \sigma < 1 \\ (A-\rho)/(1+\gamma) & \text{if } \sigma = 1 \\ (A-\rho)/\sigma & \text{if } \sigma > 1 \end{cases} \tag{A.8}$$

which is less than A in all cases. ▮

COROLLARY 1. *$\int_0^\infty ce^{-At}dt$ is finite.*

LEMMA 2. $\lim_{t\to\infty} ke^{-At} = 0$

Proof. Since $\dot{\lambda}/\lambda = (\rho - A)$, we have $\lambda(t) = \lambda(0)\mathrm{e}^{(\rho-A)t}$. Equation (5) says that $\lambda(0) > 0$. Also, the transversality condition requires $\lim_{t\to\infty} k\lambda\mathrm{e}^{-\rho t} = 0$, i.e.

$$\lim_{t\to\infty} k\lambda(0)\mathrm{e}^{(\rho-A)t}\mathrm{e}^{-\rho t} = 0.$$

Hence $\lim_{t\to\infty} k\mathrm{e}^{-At} = 0$. ▮

LEMMA 3. $k(t) = e^{At}\int_t^\infty ce^{-A\tau}d\tau$.

Proof. Equation $\dot{k} = Ak - c$ can be manipulated as follows:

$$\int_0^t (\dot{k} - Ak)\mathrm{e}^{-A\tau}d\tau = -\int_0^t c\mathrm{e}^{-A\tau}d\tau \tag{A.9}$$

$$\text{That is,}\quad k(t)e^{-At} - k(0) = \int_t^\infty c\mathrm{e}^{-A\tau}d\tau - \int_0^\infty c\mathrm{e}^{-A\tau}d\tau \tag{A.10}$$

The decomposition of the right-hand side of equation (A.9) to that of (A.10) requires that $\int_0^\infty c\mathrm{e}^{-At}dt$ be finite, which is guaranteed by the corollary above. Taking the limit as t goes to infinity in equation (A.10) and using Lemma 2, we obtain that $k(0) = \int_0^\infty c\mathrm{e}^{-A\tau}d\tau$. Thus, equation (A.10) reduces to

$$k(t) = \mathrm{e}^{At}\int_t^\infty c\mathrm{e}^{-A\tau}d\tau.$$

▮

Now we are ready to give a proof of Proposition 5.

Proof.

$$\begin{aligned}
\lim_{t\to\infty} c/k &= \lim_{t\to\infty} \frac{c}{\mathrm{e}^{At}\int_t^\infty \mathrm{ce}^{-A\tau}d\tau} \text{ (Lemma 3)} \\
&= \lim_{t\to\infty} \frac{\mathrm{ce}^{-At}}{\int_t^\infty \mathrm{ce}^{-A\tau}d\tau} \\
&= \lim_{t\to\infty} \frac{(\dot{c}-Ac)\mathrm{e}^{-At}}{-\mathrm{ce}^{-At}} \text{ by L'Hospital's Rule} \\
&= \lim_{t\to\infty}(A-\dot{c}/c) \\
&= A-\xi, \text{ (Lemma 1)}
\end{aligned}$$

which is a positive constant. ▮

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