# A Stationary Markov Equilibrium in an OLG Model

Zhixiang Zhang
CEMA, Central University of Finance and Economics
zhangzhixiang@cufe.edu.cn

Heng-fu Zou
CEMA, Central University of Finance and Economics
zouhengfu@gmail.com

**Abstract**

This paper provides sufficient conditions on the technology and preferences, under which the optimal savings-consumption policy is unique, and a stationary Markov equilibrium exists for an overlapping generation economy. Comparing with Wang (1993), our conditions on the uniqueness of optimal policy function are more general, and our conditions on the existence of equilibria depend on the exogenous parameters in the model instead of on endogenous variables.

## 1 Introduction

In an interesting paper, Wang (1993) established the existence and uniqueness of a self-fulfilling equilibrium and the existence of a nontrivial stationary equilibrium in an overlapping generations (OLG) model. The results in Wang's paper were based on a condition postulated on the endogenous function of the dynamics of capital stocks. A natural question is whether such a condition can be replaced with some restrictions on exogenous parameters. This note answers this question affirmatively. We obtain the existence of stationary equilibria under a few assumptions on the elasticity of the marginal productivity of capital. Some examples of production functions satisfying these conditions are provided.

Another extension is that for the self-fulfilling equilibrium the exogenous shocks are allowed to be from general space, not necessarily real. So we do not require any regularity condition of the technology on the shocks. Restricting the shocks to be real, we obtain the existence of nontrivial stationary equilibrium.

## 2 The Model

We follow the notations in Wang (1993), which is connected to the deterministic work of Galor and Ryder (1989) and Diamond (1965), among others. Time is discrete, $t = 1, 2, \ldots$. The population grows at rate $n > -1$. The production is affected by capital input and random shocks. The shocks, $\xi_t, t = 1, 2, \ldots$, are independent and identically distributed (i.i.d.). They are taken from an abstract measurable space, not necessarily real. At each period $t$, the per-capita production function is $f(k, \xi)$, a function of (per-capita) capital stock and random shocks. Function $f$ is assumed to satisfy the conventional properties: $f(0, \xi) = 0$ for all $\xi$, it is twice continuously differentiable, strictly concave and increasing in $k$ and $\lim_{k \to 0} f_1(k, \xi) = +\infty, \lim_{k \to \infty} f_1(k, \xi) < 1$ for all $\xi$.[1] An individual lives for two periods: he works when young and lives on his savings when old. Let $u$ and $v$ be the utility functions of these two period consumption, respectively. And they are assumed to be twice continuously differentiable, and strictly concave and increasing. At period $t$, expecting that his labor income be $\hat{w}_t$ and the return to his saving be $\hat{r}_{t+1}$, an individual chooses a consumption plan to maximize his expected life-time utility:

$$\max_{c_t^1, c_t^2} u(c_t^1) + E_t v(c_t^2) \tag{1}$$

subject to

$$c_t^1 + s_t \leq \hat{w}_t, \quad c_t^2 \leq (1 + \hat{r}_{t+1} - \delta) s_t, \tag{2}$$

where $c_t^1$ is the consumption when he is young, $c_t^2$ his consumption when old, $\delta \in (0, 1)$ is the deprecate rate of capital.

The strict monotonicity of the preference means that at the optimal the restrictions in (2) are binding, that is, the inequalities becomes equalities at optimal. From now on we will take the restrictions in (2) as equalities.

Let $r_t$ and $w_t$ be period-$t$ competitive prices for capital and labor respectively:

$$r_t = f_1(k_t, \xi_t), \tag{3}$$

$$w_t := w(k_t, \xi_t) = f(k_t, \xi_t) - k_t f_1(k_t, \xi_t). \tag{4}$$

A rational individual will make his expectation be consistent with the above relations so that $\hat{w}_t = w_t$ and $\hat{r}_t = r_t$ for all $t$. With this rational expectation and according to (3) for period $t + 1$, the Euler equation for the optimization can be written as

[1] We use the convention that the subscripts stand for derivatives, e.g., $f_1(k, \xi)$ is $\frac{\partial f}{\partial k}(k, \xi)$ and $f_{11}(k, \xi)$ is $\frac{\partial^2 f}{\partial k^2}(k, \xi)$.

$$u'(w_t - s_t) = \int_\Xi \left(1 + f_1(\frac{s_t}{1+n}, \xi_{t+1}) - \delta\right) v'\left((1 + f_1(\frac{s_t}{1+n}, \xi_{t+1}) - \delta)s_t\right) dG(\xi_{t+1}), \tag{5}$$

where $G$ is the common distribution of the shocks, $n(> -1)$ is the population growth rate, which is assumed to be constant. Note that for each time period $t$,

$$k_{t+1} = \frac{s_t}{1+n}. \tag{6}$$

Following Wang (1993), we call any $k_{t+1}$ satisfying (5) and (6) a self-fulfilling expectations equilibrium.

# 3 The Uniqueness of Self-fulfilling Expectations Equilibrium

Wang (1993, Lemma 3.1) showed that if the capital income $kf_1(k, \xi)$ is non-decreasing in $k$ almost surely, i.e.,

$$f_1(k, \xi) + kf_{11}(k, \xi) \geq 0 \quad a.e. \text{ for all } k > 0, \tag{7}$$

then, given $w_t > 0$, there is a unique self-fulfilling expectations equilibrium $k_{t+1} > 0$.

Note that the left-hand side of (5) is a strictly increasing function in $s_t$. Also note that when $s_t$ approaches $w_t$, the left-hand side approaches $+\infty$, and the right-hand side remains finite. When $s_t$ approaches zero, the left-hand side approaches some positive value, and the right-hand side tends to positive infinity.[2] Taking into account the continuity of both sides in $s_t$, a sufficient condition for a unique solution of Equation (5) is that the right-hand side is non-increasing in $s_t$, or equivalently, the right-hand side is non-increasing in $k_{t+1} = \frac{s_t}{1+n}$. Differentiating the integrand on the right-hand side with respect to $k = k_{t+1} = \frac{s_t}{1+n}$, we have

$$f_{11}v' + (1 - \delta + f_1)(1+n)v''[1 - \delta + f_1 + kf_{11}]. \tag{8}$$

It is non-positive iff

$$kf_{11} \leq \frac{-[(1 - \delta + f_1)(1+n)k]v''}{v'} \left[1 - \delta + f_1 + kf_{11}\right], \tag{9}$$

which holds iff

$$1 \geq \sigma_v \cdot \left(\frac{1 - \delta + f_1 + kf_{11}}{kf_{11}}\right), \tag{10}$$

[2]Note that $kf_1(k, \xi) < f(k, \xi) \to 0$ as $k \to 0$. Therefore, as $s_t \to 0$, the integrand $(1 + f_1(\frac{s_t}{1+n}, \xi_{t+1}) - \delta)\, v'((1 + f_1(\frac{s_t}{1+n}, \xi_{t+1}) - \delta)s_t)$ tends to $\infty \cdot v'(0) = \infty$ for all $\xi$.

where $\sigma_v$ is the Arrow-Pratt measure of relative risk aversion (RRA) (or the coefficient of relative risk aversion), which is defined as $\sigma_v(c) = -\frac{cv''(c)}{v'(c)}$ for $c > 0$. Note that the RRA is positive since $v(\cdot)$ is strictly increasing and strictly concave. Note also that $f(k, \xi)$ is strictly increasing and strictly concave in $k$. Therefore, a sufficient condition for the existence of a unique solution to Equation (5) is that the RRA does not exceed 1:

$$\sigma_v \leq 1, \quad i.e., v'(c) + cv''(c) \geq 0 \text{ for all } c > 0. \tag{11}$$

Another sufficient condtion for the existence of a unique solution to Equation (5) is,

$$1 - \delta + f_1(k, \xi) + kf_{11}(k, \xi) \geq 0 \quad a.e. \text{ for all } k > 0. \tag{12}$$

Since $\delta \in (0, 1)$, the condition in Equation (12) is weaker than Equation (7), which is required in Wang (1993, Lemma 3.1).

We summarize the arguments above as follows.

**Proposition 1.** *A sufficient condition for a unique self-fulfilling expectations equilibrium is Equation(10). In particular, either Equation (11) or Equation (12) guarantees a unique self-fulfilling expectations equilibrium.*

Condition (10) does not appear in Wang's paper, and condition (12) is obviously weaker than that in Wang (1993). Instead of Wang's condition that the capital income is nondecreasing in capital stock, our condition (12) allows that the marginal income from capital stock has a lower bound $-(1 - \delta)$ with $\delta < 1$.

Denote $s = \phi(w, \xi)$ as the solution in the above proposition. Let $\psi(k, \xi) = \phi(w(k, \xi), \xi)/(1 + n)$. We need the following lemma to prove Proposition 2 in Section 4.

**Lemma 1.** *Suppose (12) holds. Then we have (i) $0 < \phi_1(w, \xi) < 1$ for $w > 0$; (ii) $\psi_1(k, \xi) > 0$ for any $k > 0$ and all $\xi$.*

Proof. Totally differentiating (5), we have

$$u''\,[dw - ds] = E[f_{11} \frac{ds}{1+n} v' + (1 + f_1 - \delta)^2 v''\, ds + (1 + f_1 - \delta) v'' f_{11} \frac{sds}{1+n}]. \tag{13}$$

Therefore, under condition (12),

$$\frac{ds}{dw} = \frac{u''}{u'' + E[f_{11} \frac{v'}{1+n} + (1 + f_1 - \delta)^2 v'' + (1 + f_1 - \delta) v'' f_{11} \frac{s}{1+n}]} \tag{14}$$

$$= \frac{1}{1 + E[\frac{f_{11}}{u''} \frac{v'}{1+n} + (1 + f_1 - \delta)^2 \frac{v''}{u''} + (1 + f_1 - \delta) \frac{v''}{u''} \frac{s}{1+n} f_{11}]} \in (0, 1). \tag{15}$$

Notice that $w = f - kf_1$. We have

$$\frac{\partial w}{\partial k} = f_1 - f_1 - kf_{11} = -kf_{11} > 0. \tag{16}$$

By the chain rule of differentiation, $\psi_1(k, \xi) = \frac{\partial \phi}{\partial w} \frac{\partial w}{\partial k}$. Then we have (ii). *Q.E.D.*

Equation $k_{t+1} = \psi(k_t, \xi_t)$ induces a discrete time Markov process $\{k_t\}$. An invariant distribution for this Markov process is called a stationary (Markov) equilibrium of the economy. We will discuss the existence of stationary equilibria in the next section.

# 4 The Existence of a Nontrivial Stationary Equilibrium of the Economy

Obviously, the economy admits a trivial stationary equilibrium where the capital stock is always zero. More interesting is the case with a nontrivial stationary equilibrium. This is what we intend to investigate now.

Define $\mathcal{F}_0$ to be the set of deterministic production function on $R_+$ such that the elasticity of the marginal productivity of capital stock is bounded away from zero when the input is small. Formally,

$$\mathcal{F}_0 = \{g : R_+ \longrightarrow R_+ \,|\, \liminf_{k \to 0} \frac{-kg''(k)}{g'(k)} > 0\}. \tag{17}$$

Let $\mathcal{F} = \{f : R_+ \times \Xi \longrightarrow R_+ \,|\, f(\cdot, \xi) \in \mathcal{F}_0 \text{ for all } \xi\}$, the set of (random) production functions we use below.

Examples of this type of production functions include the following:

- (i) $f(k, \xi) = k^{a(\xi)}$, where $a : \Xi \to (0, 1)$ is a measurable function;
- (ii) $f(k, \xi) = a(\xi) \log k$, with $a(\cdot)$ the same as in (i);
- (iii) $f(k, \xi) = \xi g(\frac{k}{\xi})$ with $g \in \mathcal{F}_0$ and $\Xi = (0, \infty)$.

**Proposition 2.** *Suppose for all $\xi \in \Xi$, $f(\cdot, \xi) \in \mathcal{F}_0$, $\Xi$ is a compact subset of $R$, and the wage function defined by (4) is increasing in the shocks, then the economy admits at least one nontrivial stationary equilibrium.*

Proof. From (16) we have

$$\frac{\partial w}{\partial k} = -kf_{11} = \frac{-kf_{11}}{f_1} f_1. \tag{18}$$

Recall that $f_1(0,\xi) = +\infty$, as for all $\xi$ we have $f(\cdot,\xi) \in \mathcal{F}$,

$$\lim_{k \to 0} \frac{\partial w}{\partial k} = +\infty. \tag{19}$$

Therefore, we have

$$\lim_{k \to 0} \frac{\partial \psi}{\partial k} = \lim_{k \to 0} \frac{\partial \psi}{\partial w} \frac{\partial w}{\partial k} = +\infty > 1. \tag{20}$$

According to Proposition 4.3 in Wang (1993), there exists a nontrivial stationary equilibrium. *Q.E.D.*

Note that, in contrast to Wang (1993), our conditions on the existence of equilibria depend only on the exogenous parameters in the model, though we applied the result from Wang (1993).

**Acknowledgement.** M. Ali Khan and Daniel Stone provided helpful comments on an earlier version of this paper. All remaining errors are our own.

# References

[1] Diamond, Peter A., National debt in a neoclassical growth model. Ameican Economic Review 55 (1965) 1126-1150.

[2] Galor, Oded and Harl E. Ryder, Existence, uniqueness, and stability of equilibrium in an overlapping-generations model with productive capital. Journal of Economic Theory, 49:2 (1989) 360-375.

[3] Wang, Yong, Stationary equilibria in an overlapping generations economy with stochastic production. Journal of Economic Theory 61:2 (1993), 423-435.